

# THE IDEAL STRUCTURE OF ALGEBRAIC PARTIAL CROSSED PRODUCTS

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Given a partial action of a discrete group  $G$  on a Hausdorff, locally compact, totally disconnected topological space  $X$ , we consider the corresponding partial action of  $G$  on the algebra  $\mathcal{L}_c(X)$  consisting of all locally constant, compactly supported functions on  $X$ , taking values in a given field  $K$ . We then study the ideal structure of the algebraic partial crossed product  $\mathcal{L}_c(X) \rtimes G$ . After developping a theory of induced ideals, we show that every ideal in  $\mathcal{L}_c(X) \rtimes G$  may be obtained as the intersection of ideals induced from isotropy groups, thus proving an algebraic version of the Effros-Hahn conjecture.

## 1. Introduction.

The study of ideals in crossed product C\*-algebras has a long history and is best subsumed by the quest to prove and generalize the celebrated Effros-Hahn conjecture [3] formulated roughly fifty years ago. In its original form, the conjecture states that every primitive ideal in the crossed product of a commutative C\*-algebra by a locally compact group should be induced from a primitive ideal in the C\*-algebra of some isotropy group.

For the case of discrete amenable groups, the Effros-Hahn conjecture was proven by Sauvageot [9], and since then has been extended to various other contexts, notably to locally compact groups acting on non commutative C\*-algebras, as proven by Gootman and Rosenberg [6] under separability conditions. Motivated by Fack and Skandalis' study of C\*-algebras associated to foliations [10], Renault [8] realized that the Effros-Hahn conjecture also applies for groupoid C\*-algebras and proved a version of it in this context. This was later refined by Ionescu and Williams in [7].

Most treatments of the Effros-Hahn conjecture focus on the conjecture itself, namely describing a previously given ideal in the crossed product algebra in terms of induced ideals, rather than studying the relationship between the *input* ideal in the isotropy group algebra and its corresponding *output* induced ideal.

To be fair, there are a few works in the literature where this delicate relationship is discussed. Among them we should mention [15: Theorem 8.39], where a complete classification is given for the collection of primitive ideals on  $C_0(X) \rtimes G$ , where  $X$  is locally compact and  $G$  abelian. There, it is shown that every such ideal is induced from some isotropy group and hence arises from a pair  $(x, \chi)$ , where  $x$  is a point in  $X$ , and  $\chi$  a character on  $G$ . Most importantly, the primitive ideals for two pairs  $(x_1, \chi_1)$  and  $(x_2, \chi_2)$  coincide if and only if the closure of the orbit of  $x_1$  coincides with that of  $x_2$ , and  $\chi_1$  coincides with  $\chi_2$  on the isotropy group of  $x_1$ .

In the above situation, when two points have the same orbit closure, it may be shown that their isotropy groups coincide, but this relies heavily on the commutativity of  $G$ . In particular it is not even clear how to phrase the above condition in case  $G$  is not commutative.

The question of when two induced ideals coincide is also taken up in [11], where the object of study is the C\*-algebra of the Deaconu-Renault groupoid built from an action of the semigroup  $\mathbb{N}^k$  by local homeomorphisms on a locally compact space  $X$ . The primitive ideals of this C\*-algebra are shown to be parametrized by pairs  $(x, \chi)$ , where  $x$  is a point in  $X$ , and  $\chi$  a character on  $\mathbb{N}^k$ , i.e. an element of  $\mathbb{T}^k$ , and again a criterion is given for when two such pairs lead to the same primitive ideal. The condition is that the two points of  $X$  must have identical orbit closures and the two characters must coincide as characters on the interior of the isotropy of the groupoid reduced to the common orbit closure. In a sense, this result also relies on commutativity.

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In the present paper our aim is to study the Effros-Hahn conjecture in a new setting, namely the algebraic partial crossed product  $\mathcal{L}_c(X) \rtimes G$ , where  $G$  is a not necessarily commutative, discrete group partially acting on a locally compact, totally disconnected topological space  $X$ , and  $\mathcal{L}_c(X)$  is the algebra consisting of all locally constant, compactly supported functions on  $X$ , taking values in a given field  $K$ .

The justification for studying this setting comes from the current interest to investigate purely algebraic versions of some intensely studied  $C^*$ -algebras, such as the Leavitt path algebras which may be viewed as algebraic counterparts of graph  $C^*$ -algebras. In many such cases the pertinent  $C^*$ -algebra is a  $C^*$ -algebraic partial crossed product of the form  $C_0(X) \rtimes G$ , with  $X$  totally disconnected, while its algebraic sibling is the algebraic partial crossed product  $\mathcal{L}_c(X) \rtimes G$ . Steinberg algebras [12] in fact generalize this correspondence to the case where an ample étale groupoid replaces the above partial action.

One of the main results of the present paper, namely Theorem (6.3), is a version of the Effros-Hahn conjecture, where we prove that every ideal of  $\mathcal{L}_c(X) \rtimes G$  is given as the intersection of ideals induced from isotropy groups. The method of proof is entirely elementary and does not rely on the measure theoretical or analytical tools on which the main proofs in [9], [6] and [8] are based, chiefly because our setting is eminently algebraic. The strategy adopted here is as follows: given an ideal  $J$  of  $\mathcal{L}_c(X) \rtimes G$ , we first choose a representation  $\pi$  of  $\mathcal{L}_c(X) \rtimes G$  whose null space coincides with  $J$ . We then build another representation, which we call the *discretization* of  $\pi$ , whose null space coincides with that of  $\pi$ , and hence also with  $J$ . The discretized representation is then easily seen to decompose as a direct sum of sub-representations based on the orbits for the action of  $G$  on  $X$ . Each such sub-representation is finally shown to be equivalent to an induced representation, and hence the initially given ideal  $J$  is seen to coincide with the intersection of the null spaces of the various induced representation involved, each of which is then an induced ideal.

Only a tiny amount of the theory of induced ideals is necessary to prove Theorem (6.3), our version of the Effros-Hahn conjecture, but still we have chosen to start the study of induced ideals before stating and proving (6.3), mostly in order to be able to refer to the main concepts involved.

Before and after the proof of Theorem (6.3), in fact throughout the paper, we develop tools designed to understand the induction process itself, attempting to describe how exactly does an induced ideal  $Ind_{x_0}(I)$  depends on point  $x_0$  and on the ideal  $I$  it is induced from. From the outset, this dependency is expected to be quite tricky for the following reason: there are numerous examples where a crossed product algebra turns out to be simple (see e.g. [5: Theorem 4.1]), and hence the assortment of ideals in the crossed product is rather boring, but still there may be points with nontrivial isotropy (not too many since the action must be topologically free by [1], but this does not ruled out all points), and hence there may be many ideals presenting themselves as input for the induction process. However, as already mentioned, the output could be totally uninteresting due to simplicity.

The explanation for this phenomenon is that, when inducing from the isotropy group  $H_{x_0}$ , where  $x_0$  is some point in  $X$ , not all ideals in  $KH_{x_0}$  play a relevant role. Those which do, namely the ones we call *admissible*, are the only ones deserving attention in the sense that for every ideal  $I \trianglelefteq KH_{x_0}$ , there exists a unique admissible ideal  $I' \subseteq I$ , which induces the same ideal of  $\mathcal{L}_c(X) \rtimes G$  as  $I$  does. This is the content of Corollary (4.7). The correspondence  $I \mapsto Ind_{x_0}(I)$  is consequently seen to be a one-to-one mapping from the set of admissible ideals in  $KH_{x_0}$  to the set of ideals in  $\mathcal{L}_c(X) \rtimes G$ .

This naturally raises the question of which are the admissible ideals in  $KH_{x_0}$ , a question we answer in general in (4.10), and then in a few special cases in (8.4), (9.5), (9.6) and (10.3).

We then consider the question of when two induced ideals  $Ind_{x_0}(I)$  and  $Ind_{\tilde{x}_0}(\tilde{I})$  coincide, where  $x_0$  and  $\tilde{x}_0$  are two points in  $X$ ,  $I$  is an admissible ideal in  $KH_{x_0}$ , and  $\tilde{I}$  is an admissible ideal in  $KH_{\tilde{x}_0}$  (based on our study of induced ideals it suffices to consider the case where  $I$  and  $\tilde{I}$  are admissible).

As already mentioned, when  $x_0 = \tilde{x}_0$  one has that

$$Ind_{x_0}(I) = Ind_{\tilde{x}_0}(\tilde{I}) \iff I = \tilde{I},$$

so the remaining case is when  $x_0 \neq \tilde{x}_0$ . As in [15] and [11], a necessary condition for  $Ind_{x_0}(I)$  and  $Ind_{\tilde{x}_0}(\tilde{I})$  to coincide is that the orbits of  $x_0$  and  $\tilde{x}_0$  have the same closure (as long as  $I$  and  $\tilde{I}$  are proper ideals, according to (11.2)) but now, in the absence of commutativity, the isotropy groups of  $x_0$  and  $\tilde{x}_0$  no longer need to be the same, so comparing the extra data within  $KH_{x_0}$  and  $KH_{\tilde{x}_0}$  (namely  $I$  versus  $\tilde{I}$ ) is no longer a straightforward matter.

To deal with this situation we introduce the notion of *transposition* of ideals in (11.1) which is a way of comparing ideals in different isotropy groups. Our main result in that direction, namely Theorem (11.3), says that  $\text{Ind}_{x_0}(I) = \text{Ind}_{\tilde{x}_0}(\tilde{I})$  if and only if  $\tilde{I}$  is the transposition of  $I$  and vice versa.

We have already mentioned that the algebra  $\mathcal{L}_c(X) \rtimes G$ , which is our main object of study, may also be described as the Steinberg algebra for the transformation groupoid associated to the partial action of  $G$  on  $X$ . Steinberg's results obtained in [12] and [13] therefore apply to our situation as well. On the other hand, in all likelihood our results may be shown to hold for Steinberg algebras with minor modifications in our proofs.

Our algebras are all taken to be over a fixed field  $K$ , but in most places our results hold under the more general assumption that  $K$  is just a unital commutative ring. Notable exceptions are (3.16) and (8.4), where invertibility of nonzero elements in  $K$  is crucial.

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## 2. Preliminaries.

Throughout most of this work we will assume the following:

### 2.1. Standing Hypotheses.

- (a)  $K$  is a field,
- (b)  $G$  is a discrete group,
- (c)  $X$  is a Hausdorff, locally compact, totally disconnected<sup>1</sup> topological space,
- (d)  $\theta = (\{\theta_g\}_{g \in G}, \{X_g\}_{g \in G})$  is a (topological) partial action [4: Definition 5.1] of  $G$  on  $X$ , such that  $X_g$  is *clopen* (closed and open) for every  $g$  in  $G$ ,
- (e) whenever appropriate, we will also fix a distinguished point  $x_0$  in  $X$ .

Recall that a function

$$f : X \rightarrow K$$

is said to be *locally constant* if, for every  $x$  in  $X$ , there exists a neighborhood  $V$  of  $x$ , such that  $f$  is constant on  $V$ . The *support* of  $f$  is defined to be the set

$$\text{supp}(f) = \{x \in X : f(x) \neq 0\}.$$

Observe that the support of a locally constant function  $f$  is always closed, so we will not bother to define the support as the *closure* of the above set, as sometimes done in analysis.

By virtue of being locally compact and totally disconnected, we have that the topology of  $X$  admits a basis formed by *compact-open*<sup>2</sup> subsets. Given any compact-open set  $E \subseteq X$ , it is easy to see that its characteristic function, here denoted by  $1_E$ , is locally constant and compactly supported. Moreover, one may easily prove that every locally constant, compactly supported function  $f : X \rightarrow K$  is a linear combination of the form

$$f = \sum_{i=1}^n c_i 1_{E_i},$$

where the  $E_i$  are pairwise disjoint compact-open subsets and the  $c_i$  lie in  $K$ .

We will henceforth denote by  $\mathcal{L}_c(X)$  the set of all locally constant, compactly supported,  $K$ -valued functions on  $X$ . With pointwise multiplication,  $\mathcal{L}_c(X)$  is a commutative  $K$ -algebra, which is unital if and only if  $X$  is compact.

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<sup>1</sup> A locally compact topological space is *totally disconnected* if and only if it admits a basis of open sets consisting of sets which are also closed [14: Theorem 29.7].

<sup>2</sup> That is, sets which are simultaneously compact and open.

For each  $g$  in  $G$ , we may also consider the  $K$ -algebra  $\mathcal{L}_c(X_g)$ , which we will identify with the set formed by all  $f$  in  $\mathcal{L}_c(X)$  vanishing on  $X \setminus X_g$ . Under this identification  $\mathcal{L}_c(X_g)$  becomes an ideal in  $\mathcal{L}_c(X)$ .

Regarding the homomorphism  $\theta_g : X_{g^{-1}} \rightarrow X_g$ , we may define an isomorphism

$$\alpha_g : \mathcal{L}_c(X_{g^{-1}}) \rightarrow \mathcal{L}_c(X_g),$$

by setting

$$\alpha_g(f) = f \circ \theta_{g^{-1}}, \quad \forall f \in \mathcal{L}_c(X_{g^{-1}}).$$

The collection formed by all ideals  $\mathcal{L}_c(X_g)$ , together with the collection of all  $\alpha_g$ , is then easily seen to be an (algebraic) partial action [4: Definition 6.4] of  $G$  on  $\mathcal{L}_c(X)$ .

This is a *unital* partial action (one for which the domain ideals are unital) if and only if all of the  $X_g$  are compact. However we shall prefer to consider the more general situation where the  $X_g$  are only assumed to be closed (besides being open).

The main goal of this paper is to study the algebraic crossed product

$$\mathcal{L}_c(X) \rtimes G,$$

as defined in [4: Definition 8.3]. A general element  $b \in \mathcal{L}_c(X) \rtimes G$  will be denoted by

$$b = \sum_{g \in G} f_g \Delta_g,$$

where each  $f_g$  lies in  $\mathcal{L}_c(X_{g^{-1}})$ , and  $f_g = 0$ , for all but finitely many group elements  $g$ .

In many texts dealing with crossed products the above place markers “ $\Delta_g$ ” are denoted “ $\delta_g$ ”, but we shall reserve the latter to denote elements in  $KG$ , such as

$$\sum_{h \in H} c_h \delta_h,$$

where the  $c_h$  are scalars in  $K$ , again equal to zero except for finitely many group elements  $g$ . In fact, throughout this paper all summations will be finite, either because the set of indices is finite, or because all but finitely many summands are supposed to vanish.

Since we are assuming that every  $X_g$  is clopen, its characteristic function  $1_{X_g}$ , which we will abbreviate to

$$1_g := 1_{X_g},$$

is a locally constant function, although not necessarily compactly supported. However, given any  $f$  in  $\mathcal{L}_c(X)$ , one has that  $f 1_{g^{-1}}$  is compactly supported, so it belongs to  $\mathcal{L}_c(X_g)$ . We may therefore define

$$\bar{\alpha}_g(f) := \alpha_g(f 1_{g^{-1}}).$$

so that  $\bar{\alpha}_g$  is a globally defined endomorphism of  $\mathcal{L}_c(X)$ .

Recall that if  $g$  and  $h$  are elements of  $G$ , and if  $e \in \mathcal{L}_c(X_g)$ , and  $f \in \mathcal{L}_c(X_h)$ , then the product of  $e \Delta_g$  by  $f \Delta_h$  is defined by

$$(e \Delta_g)(f \Delta_h) = \alpha_g(\alpha_{g^{-1}}(e) f) \Delta_{gh}. \quad (2.2)$$

In our present situation this expression may be made simpler as follows: since  $\alpha_{g^{-1}}(e)$  lies in  $\mathcal{L}_c(X_{g^{-1}})$ , we have that

$$\alpha_{g^{-1}}(e) = \alpha_{g^{-1}}(e) 1_{g^{-1}},$$

so the coefficient of  $\Delta_{gh}$  in (2.2) equals

$$\alpha_g(\alpha_{g^{-1}}(e) f) = \alpha_g(\alpha_{g^{-1}}(e) 1_{g^{-1}} f) = \alpha_g(\alpha_{g^{-1}}(e)) \alpha_g(1_{g^{-1}} f) = e \bar{\alpha}_g(f).$$

The promised simpler formula for the product thus reads

$$(e \Delta_g)(f \Delta_h) = e \bar{\alpha}_g(f) \Delta_{gh}. \quad (2.3)$$

### 3. Induction.

As always, we assume the conditions set out in (2.1). From here on the distinguished point  $x_0$  mentioned in (2.1.e) will become important in our development and we will henceforth use the following notations

$$\begin{aligned}\mathcal{S}_{x_0} &:= \{g \in G : x_0 \in X_{g^{-1}}\}, \\ H_{x_0} &:= \{g \in G : x_0 \in X_{g^{-1}}, \theta_g(x_0) = x_0\}, \\ \text{Orb}(x_0) &:= \{\theta_g(x_0) : g \in \mathcal{S}\}.\end{aligned}\tag{3.1}$$

Whenever there is no question as to which point  $x_0$  we are referring to, as it will often be the case, we will omit subscripts and write  $\mathcal{S}$  and  $H$  in place of  $\mathcal{S}_{x_0}$  and  $H_{x_0}$ , respectively.

Notice that  $H$  is a subgroup of  $G$ , often called the *isotropy* group of  $x_0$ . On the other hand observe that  $\mathcal{S}H \subseteq \mathcal{S}$ , so  $\mathcal{S}$  is a union of left  $H$ -classes.

The map

$$g \in \mathcal{S} \mapsto \theta_g(x_0) \in \text{Orb}(x_0)$$

is clearly onto, and two elements  $g_1$  and  $g_2$  in  $\mathcal{S}$  satisfy  $\theta_{g_1}(x_0) = \theta_{g_2}(x_0)$ , if and only if they lie in the same left  $H$ -class.

A central ingredient in the induction process to be introduced shortly, is the subspace  $M$  of the group algebra  $KG$  given by

$$M = \text{span}\{\delta_g : g \in \mathcal{S}\}.$$

As already observed,  $\mathcal{S}H \subseteq \mathcal{S}$ , so it follows that  $M$  is naturally a right  $KH$ -module.

Consider the unique bilinear form

$$\langle \cdot, \cdot \rangle : M \times M \rightarrow KH$$

such that

$$\langle \delta_k, \delta_l \rangle = \begin{cases} \delta_{k^{-1}l}, & \text{if } k^{-1}l \in H, \\ 0, & \text{otherwise.} \end{cases}\tag{3.2}$$

This may also be written as

$$\langle \delta_k, \delta_l \rangle = [k^{-1}l \in H] \delta_{k^{-1}l},$$

where the brackets indicate *boolean value*<sup>3</sup>.

An important property of this form is expressed by the identity

$$\langle m, na \rangle = \langle m, n \rangle a, \quad \forall m, n \in M, \quad \forall a \in KH,\tag{3.3}$$

which the reader may easily prove.

**3.4. Proposition.** *If  $R \subseteq \mathcal{S}$  is a system of representatives of left  $H$ -classes, so that*

$$\mathcal{S} = \dot{\bigcup}_{r \in R} rH,$$

*then, for all  $m$  in  $M$ , one has*

$$m = \sum_{r \in R} \delta_r \langle \delta_r, m \rangle,$$

*where the sum is always finite in the sense that there are only finitely many nonzero summands.*

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<sup>3</sup> In fact we shall often use boolean values in this work, sometimes in a slightly abusive fashion, such as in

$$[x \in X_{g^{-1}}] f(\theta_g(x)),$$

where  $f$  is some scalar valued function on  $X$ . The principle behind this is that, when  $x$  is not in the domain  $X_{g^{-1}}$  of  $\theta_g$ , so that  $\theta_g(x)$  is not defined, the zero boolean value of the expression “ $x \in X_{g^{-1}}$ ” predominates and turns the whole expression into zero. In other words, *zero* times something which is *not defined* is taken to be zero. It is true that an excessive abuse of this principle may perhaps lead to unexpected consequences, but we promise to use it only to shorten expressions which could otherwise be written in two clauses, such as (3.2).

*Proof.* Assuming that  $m = \delta_l$ , there exists a unique  $k$  in  $R$  such that  $lH = kH$ , which is to say that  $k^{-1}l \in H$ . Then

$$\sum_{r \in R} \delta_r \langle \delta_r, \delta_l \rangle = \sum_{r \in R} \delta_r [r^{-1}l \in H] \delta_{r^{-1}l} = \delta_k \delta_{k^{-1}l} = \delta_l.$$

The general case follows by writing  $m$  as a linear combination of the  $\delta_l$ . □

Besides being a right  $KH$ -module,  $M$  is also a left module:

**3.5. Proposition.** *There is a left  $(\mathcal{L}_c(X) \rtimes G)$ -module structure on  $M$  such that*

$$(f \Delta_g) \delta_l = [gl \in \mathcal{S}] f(\theta_{gl}(x_0)) \delta_{gl},$$

for every  $f \in \mathcal{L}_c(X_g)$ , and all  $l \in \mathcal{S}$ . With this,  $M$  moreover becomes an  $(\mathcal{L}_c(X) \rtimes G)$  -  $KH$  - bimodule.

*Proof.* Left for the reader. □

Given any left  $KH$ -module  $V$ , one may therefore build the left  $(\mathcal{L}_c(X) \rtimes G)$ -module

$$M \otimes_{KH} V,$$

henceforth denoted simply by  $M \otimes V$ . This left module structure is well known, but it might be worth spelling it out here: given  $b \in \mathcal{L}_c(X) \rtimes G$ , one has

$$b(m \otimes v) = (bm) \otimes v, \quad \forall m \in M, v \in V.$$

**3.6. Definition.** The left  $(\mathcal{L}_c(X) \rtimes G)$ -module  $M \otimes V$  mentioned above is said to be the module *induced* by  $V$ .

Recall that a module  $V$  is said to be *irreducible* if it has no nontrivial submodules or, equivalently, if the submodule generated by any nonzero element coincides with  $V$ .

**3.7. Proposition.** *If  $V$  is an irreducible  $KH$ -module, then  $M \otimes V$  is irreducible as a  $(\mathcal{L}_c(X) \rtimes G)$ -module.*

*Proof.* Given any nonzero vector  $w \in M \otimes V$ , we must show that the submodule it generates, here denoted by  $\langle w \rangle$ , coincides with  $M \otimes V$ . In order to do this, write

$$w = \sum_{i=1}^n m_i \otimes u_i,$$

and let  $R \subseteq \mathcal{S}$  be a system of representatives of left classes for  $\mathcal{S}$  modulo  $H$ . So by (3.4) we have

$$w = \sum_{i=1}^n \sum_{r \in R} \delta_r \langle \delta_r, m_i \rangle \otimes u_i = \sum_{r \in R} \sum_{i=1}^n \delta_r \otimes \langle \delta_r, m_i \rangle u_i = \sum_{r \in R} \delta_r \otimes \sum_{i=1}^n \langle \delta_r, m_i \rangle u_i = \sum_{r \in R} \delta_r \otimes v_r,$$

where the  $v_r$  are defined by the last equality above. Since all sums involved are finite, the set

$$\Gamma = \{r \in R : v_r \neq 0\}$$

must be finite. It is moreover nonempty, since we are assuming that  $w \neq 0$ .

Fixing any  $s$  in  $R$ , we claim that  $\delta_s \otimes v_s$  lies in  $\langle w \rangle$ . To see this, notice that no two elements of  $R$  are in the same left  $H$ -class, so the points  $\theta_r(x_0)$  are pairwise distinct. We may then pick some  $f$  in  $\mathcal{L}_c(X)$  such that  $f(\theta_s(x_0)) = 1$ , while  $f(\theta_r(x_0)) = 0$ , for all  $r \in \Gamma \setminus \{s\}$ . We then have

$$\langle w \rangle \ni fw = \sum_{r \in \Gamma} f \delta_r \otimes v_r = \sum_{r \in \Gamma} [r \in \mathcal{S}] f(\theta_r(x_0)) \delta_r \otimes v_r = \delta_s \otimes v_s.$$

We next show that  $\langle w \rangle$  contains  $\delta_s \otimes V$ . Since  $V$  is irreducible as a  $KH$ -module, we have that  $V$  is spanned by the set  $\{\delta_h v_s : h \in H\}$ , so it is enough to prove that

$$\delta_s \otimes \delta_h v_s \in \langle w \rangle, \quad \forall h \in H. \quad (3.7.1)$$

We thus fix some  $h$  in  $H$ , and put  $g = shs^{-1}$ . Observing that

$$\theta_s(x_0) = \theta_s(\theta_h(x_0)) = \theta_{sh}(x_0) = \theta_{gs}(x_0),$$

we see that  $x_0 \in X_{(gs)^{-1}} \cap X_{s^{-1}}$ . Consequently

$$\theta_{gs}(x_0) \in \theta_{gs}(X_{(gs)^{-1}} \cap X_{s^{-1}}) = X_{gs} \cap X_g.$$

We may then choose some  $f$  in  $\mathcal{L}_c(X_g)$  such that  $f(\theta_{gs}(x_0)) = 1$ , and then

$$\langle w \rangle \ni f \Delta_g(\delta_s \otimes v_s) = [gs \in \mathcal{S}] f(\theta_{gs}(x_0)) \delta_{gs} \otimes v_s = \delta_{gs} \otimes v_s = \delta_{sh} \otimes v_s = \delta_s \delta_h \otimes v_s = \delta_s \otimes \delta_h v_s,$$

thus proving (3.7.1), and hence that  $\delta_s \otimes V \subseteq \langle w \rangle$ .

We will conclude the proof by showing that  $\delta_k \otimes V \subseteq \langle w \rangle$ , for every  $k$  in  $\mathcal{S}$ . Given any such  $k$ , set  $g = ks^{-1}$ , and notice that  $x_0 \in X_{k^{-1}} \cap X_{s^{-1}}$ , so

$$\theta_k(x_0) \in \theta_k(X_{k^{-1}} \cap X_{s^{-1}}) = X_k \cap X_{ks^{-1}} \subseteq X_g.$$

So we may find some  $f$  in  $\mathcal{L}_c(X_g)$  such that  $f(\theta_k(x_0)) = 1$ , and for every  $v$  in  $V$ , one has

$$\langle w \rangle \ni f \Delta_g(\delta_s \otimes v) = [gs \in \mathcal{S}] f(\theta_{gs}(x_0)) \delta_{gs} \otimes v = [k \in \mathcal{S}] f(\theta_k(x_0)) \delta_k \otimes v = \delta_k \otimes v.$$

This shows that  $\delta_k \otimes V \subseteq \langle w \rangle$ , as desired, and hence that  $\langle w \rangle = M \otimes V$ , concluding the proof.  $\square$

Our next goal will be to compute the annihilator of the induced module in terms of the annihilator of the original module  $V$ . We begin with a useful technical result.

**3.8. Lemma.** *Let  $V$  be a left  $KH$ -module and let  $I$  be the annihilator of  $V$  in  $KH$ . Given  $m \in M$ , the following are equivalent:*

- (i)  $m \otimes v = 0$ , for all  $v$  in  $V$ ,
- (ii)  $\langle n, m \rangle \in I$ , for all  $n \in M$ .

*Proof.* (ii)  $\Rightarrow$  (i): Let  $R$  be a system of representatives of left  $H$ -classes in  $\mathcal{S}$ . Then for every  $v$  in  $V$  we have

$$m \otimes v \stackrel{(3.4)}{=} \sum_{r \in R} \delta_r \langle \delta_r, m \rangle \otimes v = \sum_{r \in R} \delta_r \otimes \langle \delta_r, m \rangle v = 0.$$

(i)  $\Rightarrow$  (ii): Fixing  $n \in M$ , consider the bilinear mapping

$$(m, v) \in M \times V \mapsto \langle n, m \rangle v \in V.$$

By (3.3) this is  $KH$ -balanced, so there is a well defined  $K$ -linear mapping  $T_n : M \otimes V \rightarrow V$ , such that

$$T_n(m \otimes v) = \langle n, m \rangle v,$$

for all  $m$  in  $M$ , and all  $v$  in  $V$ . Assuming that  $m$  satisfies (i) we then have that

$$\langle n, m \rangle v = T_n(m \otimes v) = 0, \quad \forall n \in M, \quad \forall v \in V,$$

so  $\langle n, m \rangle$  annihilates  $V$ , whence  $\langle n, m \rangle \in I$ , proving (ii).  $\square$

As a consequence we obtain the following description of the annihilator of an induced module.

**3.9. Corollary.** *Let  $V$  be a left  $KH$ -module and let  $I$  be the annihilator of  $V$  in  $KH$ . Then the annihilator of  $M \otimes V$  in  $\mathcal{L}_c(X) \rtimes G$  is given by*

$$\{b \in \mathcal{L}_c(X) \rtimes G : \langle n, bm \rangle \in I, \forall n, m \in M\}.$$

*In particular the annihilator of  $M \otimes V$  depends only on  $I$ .*

*Proof.* One has that  $b$  lies in the annihilator of  $M \otimes V$ , iff  $bm \otimes v = 0$ , for all  $m$  and  $v$ , which is equivalent to saying that  $\langle n, bm \rangle \in I$ , for all  $n$  and  $m$ , by (3.8).  $\square$

Since the annihilator of  $M \otimes V$  depends only on  $I$ , rather than on  $V$ , we may think of it as built out of  $I$ . To account for this we give the following:

**3.10. Definition.** Given any ideal<sup>4</sup>  $I \trianglelefteq KH$ , we shall let

$$Ind_{x_0}(I) := \{b \in \mathcal{L}_c(X) \rtimes G : \langle n, bm \rangle \in I, \forall n, m \in M\}.$$

This will be referred to as the *ideal induced by  $I$* . When there is no risk of confusion we shall write this simply as  $Ind(I)$ .

Reinterpreting (3.9) with the terminology just introduced, we have:

**3.11. Proposition.** *Let  $V$  be a left  $KH$ -module and let  $I$  be the annihilator of  $V$  in  $KH$ . Then the annihilator of  $M \otimes V$  coincides with the ideal induced by  $I$ .*

So far it is clear that  $Ind(I)$  is a *right* ideal, but we will shortly prove that it is indeed a two-sided ideal. The behavior of the induction process under inclusion and intersection is easy to understand:

**3.12. Proposition.**

- (i) *If  $I_1$  and  $I_2$  are ideals of  $KH$  with  $I_1 \subseteq I_2$ , then  $Ind(I_1) \subseteq Ind(I_2)$ .*
- (ii) *Given any family  $\{I_\lambda\}_{\lambda \in \Lambda}$  of ideals of  $KH$ , then  $Ind\left(\bigcap_{\lambda \in \Lambda} I_\lambda\right) = \bigcap_{\lambda \in \Lambda} Ind(I_\lambda)$ .*

*Proof.* Follows easily by inspecting the definitions involved.  $\square$

When checking that  $\langle n, bm \rangle \in I$ , for all  $n, m \in M$ , as required by the above definition, it suffices to consider  $m = \delta_k$  and  $n = \delta_l$ , for  $k, l \in \mathcal{S}$ , since these generate  $M$ . It is therefore nice to have an explicit formula for use in this situation:

**3.13. Proposition.** *Given  $b = \sum_{g \in G} f_g \Delta_g$  in  $\mathcal{L}_c(X) \rtimes G$ , and given  $k$  and  $l$  in  $\mathcal{S}$ , one has that*

$$\langle \delta_k, b\delta_l \rangle = \sum_{g \in kHl^{-1}} f_g(\theta_k(x_0)) \delta_{k^{-1}gl}.$$

*Proof.* We have

$$\begin{aligned} \langle \delta_k, b\delta_l \rangle &= \sum_{g \in G} \langle \delta_k, (f_g \Delta_g) \delta_l \rangle = \sum_{g \in G} \langle \delta_k, [gl \in \mathcal{S}] f_g(\theta_{gl}(x_0)) \delta_{gl} \rangle = \\ &= \sum_{g \in kHl^{-1}} [gl \in \mathcal{S}] f_g(\theta_{gl}(x_0)) \delta_{k^{-1}gl} = \dots \end{aligned}$$

If  $g \in kHl^{-1}$ , then  $gl$  lies in  $kH$ , so  $\theta_{gl}(x_0)$  is indeed defined, meaning that  $gl \in \mathcal{S}$ , and  $\theta_{gl}(x_0)$  coincides with  $\theta_k(x_0)$ . So the above equals

$$\dots = \sum_{g \in kHl^{-1}} f_g(\theta_k(x_0)) \delta_{k^{-1}gl},$$

concluding the proof.  $\square$

The above computation allows for a very concrete criteria for membership in  $Ind(I)$ , namely:

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<sup>4</sup> Unless otherwise stated, all ideals in this paper are assumed to be two-sided.



**3.14. Proposition.** *Given any ideal  $I \trianglelefteq KH$ , and given any  $b = \sum_{g \in G} f_g \Delta_g$  in  $\mathcal{L}_c(X) \rtimes G$ , one has that  $b \in \text{Ind}(I)$ , if and only if, for every  $k$  and  $l$  in  $\mathcal{S}$ , one has that*

$$\sum_{g \in kHl^{-1}} f_g(\theta_k(x_0)) \delta_{k^{-1}gl} \in I.$$

*Proof.* Follows from (3.13) and the fact that the  $\delta_k$  generate  $M$ , as a  $K$ -vector space.  $\square$

Let us now discuss two trivial examples:

**3.15. Proposition.**

- (a) *If  $I = KH$ , then  $\text{Ind}(I)$  coincides with the whole algebra  $\mathcal{L}_c(X) \rtimes G$ .*
- (b) *If  $I = \{0\}$ , then*

$$\text{Ind}(I) = \left\{ \sum_{g \in G} f_g \Delta_g \in \mathcal{L}_c(X) \rtimes G : f_g|_{\overline{\text{Orb}(x_0)}} = 0, \forall g \in G \right\}.$$

*Proof.* The first statement is clear. As for (b), first notice that a locally constant function vanishing on  $\text{Orb}(x_0)$ , necessarily also vanishes on the closure  $\overline{\text{Orb}(x_0)}$ . This said, let

$$b = \sum_{g \in G} f_g \Delta_g \in \mathcal{L}_c(X) \rtimes G.$$

Assuming that  $b$  lies in  $\text{Ind}(I)$ , and given any point  $y$  in the orbit of  $x_0$ , we will prove that  $f_g(y) = 0$ , for all  $g$ . In case  $y \notin X_g$ , it is clear that  $f_g(y) = 0$ , since the support of  $f_g$  is contained in  $X_g$ . Otherwise, if  $y \in X_g$ , write  $y = \theta_k(x_0)$ , for some  $k \in \mathcal{S}$ , and observe that  $y \in X_k \cap X_g$ , so

$$x_0 = \theta_{k^{-1}}(y) \in \theta_{k^{-1}}(X_k \cap X_g) = X_{k^{-1}} \cap X_{k^{-1}g},$$

from where we see that  $l := g^{-1}k$  lies in  $\mathcal{S}$ . Consequently

$$\{0\} = I \ni \langle \delta_k, b \delta_l \rangle \stackrel{(3.13)}{=} \sum_{g' \in kHl^{-1}} f_{g'}(\theta_k(x_0)) \delta_{k^{-1}g'l}.$$

Among the above summands, one is to find  $g' = kl^{-1} = kk^{-1}g = g$ , so in particular

$$0 = f_{g'}(\theta_k(x_0)) = f_g(y).$$

This shows that  $f_g$  vanishes on the orbit of  $x_0$ , and hence also on its closure.

Conversely, assuming that each  $f_g$  vanishes on the orbit of  $x_0$ , it is clear from (3.13) that  $\langle \delta_k, b \delta_l \rangle = 0 \in I$ , so  $b \in \text{Ind}(I)$ .  $\square$

Much has been said about the intersection of an ideal in a crossed product algebra and its intersection with the coefficient algebra. In the case of induced ideals we have:

**3.16. Proposition.** *Let  $I$  be a proper<sup>5</sup> ideal in  $KH$ . Then the intersection  $\text{Ind}(I) \cap \mathcal{L}_c(X)$  consists of all  $f$  in  $\mathcal{L}_c(X)$  vanishing on  $\overline{\text{Orb}(x_0)}$ .*

*Proof.* Let  $f \in \text{Ind}(I) \cap \mathcal{L}_c(X)$ . Then, choosing any  $k$  in  $\mathcal{S}$ , we have

$$I \ni \langle \delta_k, f \delta_k \rangle \stackrel{(3.13)}{=} f(\theta_k(x_0)) \delta_1.$$

Should  $f(\theta_k(x_0))$  not vanish, the above would be an invertible element in  $I$ , whence  $I = KH$ , contradicting the hypothesis. Thus  $f(\theta_k(x_0)) = 0$ , showing that  $f$  vanishes on the orbit of  $x_0$ , and hence also on its closure.

Conversely, if  $f$  vanishes on  $\overline{\text{Orb}(x_0)}$ , then by (3.15.ii)

$$f \in \text{Ind}(\{0\}) \subseteq \text{Ind}(I). \quad \square$$

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<sup>5</sup> We say that an ideal in an algebra is proper when it is not equal to the whole algebra.

Consider the map

$$E = E_{x_0} : \mathcal{L}_c(X) \rtimes G \rightarrow \mathcal{L}_c(X) \rtimes H, \quad (3.17)$$

given by

$$E\left(\sum_{g \in G} f_g \Delta_g\right) = \sum_{h \in H} f_h \Delta_h.$$

This is sometimes called a *conditional expectation*. One of its important properties is that of being a  $(\mathcal{L}_c(X) \rtimes H)$ -bimodule map in the sense that if  $a \in \mathcal{L}_c(X) \rtimes H$ , and  $b \in \mathcal{L}_c(X) \rtimes G$ , then

$$E(ab) = aE(b) \quad \text{and} \quad E(ba) = E(b)a. \quad (3.18)$$

It is also evident that  $E$  is a projection from  $\mathcal{L}_c(X) \rtimes G$  onto  $\mathcal{L}_c(X) \rtimes H$ .

Consider also the map

$$\nu = \nu_{x_0} : \mathcal{L}_c(X) \rtimes H \rightarrow KH, \quad (3.19)$$

given by

$$\nu\left(\sum_{h \in H} f_h \Delta_h\right) = \sum_{h \in H} f_h(x_0) \delta_h.$$

Since  $x_0$  is fixed by  $H$ , one may easily show that  $\nu$  is an algebra homomorphism.

The composition of  $\nu$  and  $E$  is therefore the map

$$F = F_{x_0} = \nu \circ E : \mathcal{L}_c(X) \rtimes G \rightarrow KH, \quad (3.20)$$

given by

$$F\left(\sum_{g \in G} f_g \Delta_g\right) = \sum_{h \in H} f_h(x_0) \delta_h.$$

There is a useful relationship between  $F$  and the above bilinear form  $\langle \cdot, \cdot \rangle$ , expressed as follows:

**3.21. Lemma.** *Let  $k, l \in G$ , and choose  $p \in \mathcal{L}_c(X_{k^{-1}})$ , and  $q \in \mathcal{L}_c(X_l)$ , so that defining*

$$u = p \Delta_{k^{-1}}, \quad \text{and} \quad v = q \Delta_l,$$

*one has that  $u$  and  $v$  are in  $\mathcal{L}_c(X) \rtimes G$ . Then, for every  $b$  in  $\mathcal{L}_c(X) \rtimes G$ , one has that*

$$F(uv) = \begin{cases} p(x_0) q(\theta_l(x_0)) \langle \delta_k, b \delta_l \rangle, & \text{if } k, l \in \mathcal{S}, \\ 0, & \text{otherwise.} \end{cases}$$

*Proof.* Write  $b = \sum_{g \in G} f_g \Delta_g$ , so that

$$\begin{aligned} E(uv) &= E\left(\sum_{g \in G} p \Delta_{k^{-1}} f_g \Delta_g q \Delta_l\right) = E\left(\sum_{g \in G} p \alpha_{k^{-1}}(f_g 1_k) \alpha_{k^{-1}g}(q 1_{g^{-1}k}) \Delta_{k^{-1}gl}\right) = \\ &= \sum_{g \in kHl^{-1}} p \alpha_{k^{-1}}(f_g 1_k) \alpha_{k^{-1}g}(q 1_{g^{-1}k}) \Delta_{k^{-1}gl}. \end{aligned}$$

Therefore

$$\begin{aligned} F(uv) &= \nu(E(uv)) = \sum_{g \in kHl^{-1}} p(x_0) \alpha_{k^{-1}}(f_g 1_k)|_{x_0} \alpha_{k^{-1}g}(q 1_{g^{-1}k})|_{x_0} \delta_{k^{-1}gl} = \\ &= \sum_{g \in kHl^{-1}} p(x_0) [x_0 \in X_{k^{-1}}] f_g(\theta_k(x_0)) [x_0 \in X_{k^{-1}g}] q(\theta_{g^{-1}k}(x_0)) \delta_{k^{-1}gl} = \dots \end{aligned}$$

Notice that, whenever  $r^{-1}s \in H$ , one has that  $\theta_{r^{-1}s}(x_0) = x_0$ , so

$$x_0 \in X_{r^{-1}} \iff x_0 \in X_{s^{-1}},$$

and if these equivalent conditions hold then

$$\theta_r(x_0) = \theta_s(x_0).$$

Applying this to  $r = g^{-1}k$ , and  $s = l$ , we see that the above equals

$$\begin{aligned} \cdots &= \sum_{g \in kHl^{-1}} p(x_0) [x_0 \in X_{k-1}] f_g(\theta_k(x_0)) [x_0 \in X_{l-1}] q(\theta_l(x_0)) \delta_{k^{-1}gl} = \\ &= [x_0 \in X_{k-1}] [x_0 \in X_{l-1}] p(x_0) q(\theta_l(x_0)) \sum_{g \in kHl^{-1}} f_g(\theta_k(x_0)) \delta_{k^{-1}gl} \stackrel{(3.13)}{=} [k, l \in \mathcal{S}] p(x_0) q(\theta_l(x_0)) \langle \delta_k, b\delta_l \rangle. \quad \square \end{aligned}$$

Let us now use the above result with the purpose of giving an alternative definition of  $\text{Ind}(I)$ , where  $F$  is employed instead of the form  $\langle \cdot, \cdot \rangle$ .

**3.22. Proposition.** *Given any ideal  $I \trianglelefteq KH$ , one has that*

$$\text{Ind}(I) = \{b \in \mathcal{L}_c(X) \rtimes G : F(ubv) \in I, \forall u, v \in \mathcal{L}_c(X) \rtimes G\}.$$

*Proof.* In order to prove the statement we must show that, for any given  $b \in \mathcal{L}_c(X) \rtimes G$ , the following are equivalent:

- (i)  $F(ubv) \in I$ , for all  $u, v \in \mathcal{L}_c(X) \rtimes G$ ,
- (ii)  $\langle n, bm \rangle \in I$ , for all  $n, m \in M$ .

(i)  $\Rightarrow$  (ii): It is clearly enough to prove (ii) for  $n = \delta_k$ , and  $m = \delta_l$ , where  $k$  and  $l$  are arbitrary elements of  $\mathcal{S}$ . In order to do this, pick  $p \in \mathcal{L}_c(X_{k-1})$ , and  $q \in \mathcal{L}_c(X_l)$ , such that  $p(x_0) = 1$ , and  $q(\theta_l(x_0)) = 1$ . Then

$$\langle n, bm \rangle = \langle \delta_k, b\delta_l \rangle = p(x_0) q(\theta_l(x_0)) \langle \delta_k, b\delta_l \rangle \stackrel{(3.21)}{=} F(ubv) \in I,$$

proving (ii).

(ii)  $\Rightarrow$  (i): It is clearly enough to prove (i) for  $u = p\Delta_{k-1}$ , and  $v = q\Delta_l$ , where  $k$  and  $l$  are arbitrary elements of  $G$ , while  $p \in \mathcal{L}_c(X_{k-1})$ , and  $q \in \mathcal{L}_c(X_l)$ . If  $k$  and  $l$  lie in  $\mathcal{S}$ , we have that

$$F(ubv) \stackrel{(3.21)}{=} p(x_0) q(\theta_l(x_0)) \langle \delta_k, b\delta_l \rangle \in I.$$

On the other hand, if either  $k$  or  $l$  are not in  $\mathcal{S}$ , then again by (3.21), we have that  $F(ubv) = 0 \in I$ .  $\square$

When  $I$  is the annihilator of a left  $KH$ -module  $V$ , we have seen that  $\text{Ind}(I)$  is the annihilator of  $M \otimes V$ , hence a two-sided ideal. Should there be any doubt that  $\text{Ind}(I)$  is always a two-sided ideal (in case  $I$  is not presented<sup>6</sup> as the annihilator of some left  $KH$ -module), the above description of  $\text{Ind}(I)$  may be used to dispell this doubt.

#### 4. Admissible ideals.

As always we assume (2.1). So far we have not considered the question of which ideals  $I \trianglelefteq KH$  actually lead to nontrivial induced ideals. In case  $\theta$  is topologically free and minimal, a situation well known to lead to a simple crossed product [5: Theorem 4.1], at least some points  $x_0$  in  $X$  are allowed to possess a nontrivial isotropy group  $H$ , and hence there might be plenty ideals  $I \trianglelefteq KH$  to choose from, but the simplicity of  $\mathcal{L}_c(X) \rtimes G$  prevents induced ideals from being nontrivial.

In this section we shall begin to explore the delicate relationship between ideals of the isotropy group algebra and the induced ideals they lead to.

The map  $F_{x_0}$ , introduced in (3.20), will play a crucial role in this section. Because we will always consider the induction process relative to the point  $x_0$  fixed in (2.1.e), we will abolish the subscript writing  $F$  in place of  $F_{x_0}$ .

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<sup>6</sup> Notice, however, that any ideal of a unital algebra is the annihilator of some left module, namely the quotient algebra.

**4.1. Proposition.** *Given  $a$  and  $b$  in  $\mathcal{L}_c(X) \rtimes G$ , such that either  $a$  or  $b$  is in  $\mathcal{L}_c(X) \rtimes H$ , then*

$$F(ab) = F(a)F(b).$$

*Proof.* Suppose that  $b$  is in  $\mathcal{L}_c(X) \rtimes H$ . Then, using (3.18), we have

$$F(ab) = \nu(E(ab)) = \nu(E(a)b) = \nu(E(a)E(b)) = \nu(E(a))\nu(E(b)) = F(a)F(b).$$

A similar reasoning applies when  $a$  is in  $\mathcal{L}_c(X) \rtimes H$ . □

In particular, if  $\varphi$  is in  $\mathcal{L}_c(X)$ , then

$$F(a\varphi) = \varphi(x_0)F(a) = F(\varphi a), \quad \forall a \in \mathcal{L}_c(X) \rtimes G. \quad (4.2)$$

Another instance of (4.1) is obtained when  $\varphi$  is supported on  $X_h$ , for some  $h$  in  $H$ , in which case we have

$$F(a\varphi\Delta_h) = F(a)\varphi(x_0)\delta_h, \quad \text{and} \quad F(\varphi\Delta_h a) = \varphi(x_0)\delta_h F(a). \quad (4.3)$$

**4.4. Proposition.** *If  $J$  is any ideal in  $\mathcal{L}_c(X) \rtimes G$ , then  $F(J)$  is an ideal in  $KH$ .*

*Proof.* Given  $c \in F(J)$ , and  $d \in KH$ , we must prove that  $cd$  and  $dc$  lie in  $F(J)$ , and it clearly suffices to assume that  $d = \delta_h$ , for some  $h$  in  $H$ . Choose  $b$  in  $J$  such that  $F(b) = c$ , and let  $\varphi \in \mathcal{L}_c(X_h)$  be such that  $\varphi(x_0) = 1$ . Then

$$cd = c\varphi(x_0)\delta_h = F(b)F(\varphi\Delta_h) \stackrel{(4.3)}{=} F(b\varphi\Delta_h) \in F(J).$$

A similar reasoning proves that  $dc \in F(J)$ . □

Applying this to induced ideals we get the following:

**4.5. Proposition.** *Let  $I$  be an ideal in  $KH$ , and put  $I' = F(\text{Ind}(I))$ . Then*

- (i)  $I'$  is an ideal of  $KH$ ,
- (ii)  $I' \subseteq I$ ,
- (iii)  $\text{Ind}(I') = \text{Ind}(I)$ .

*Proof.* (i) Follows from (4.4).

(ii) Given  $a$  in  $I'$ , write  $a = F(b)$ , with  $b \in \text{Ind}(I)$ . Choosing  $u \in \mathcal{L}_c(X)$ , with  $u(x_0) = 1$ , we then have

$$I \stackrel{(3.22)}{\ni} F(ubu) \stackrel{(4.2)}{=} u(x_0)F(b)u(x_0) = F(b) = a.$$

This proves (ii).

(iii) Since  $I' \subseteq I$ , it is obvious that  $\text{Ind}(I') \subseteq \text{Ind}(I)$ . On the other hand, if  $b \in \text{Ind}(I)$ , then for every  $u$  and  $v$  in  $\mathcal{L}_c(X) \rtimes G$ , one has that  $ubv \in \text{Ind}(I)$ , whence

$$F(ubv) \in F(\text{Ind}(I)) = I'.$$

This proves that  $b \in \text{Ind}(I')$ , concluding the proof. □

The grand conclusion of this result is that, should we want to catalogue all induced ideals, we do not need to consider all ideals  $I \trianglelefteq KH$ , since  $I$  may be replaced by  $I'$ , without affecting the outcome of the induction process.

This motivates the question of how to separate the ideals that matter from those which don't, a task we now begin to undertake.

**4.6. Definition.** An ideal  $I \trianglelefteq KH$  is said to be *admissible* if  $F(\text{Ind}(I)) = I$ .

Interpreting (4.5) from the point of view of the concept just introduced we have:

**4.7. Corollary.** *For every ideal  $I \trianglelefteq KH$ , there exists a unique admissible ideal  $I' \subseteq I$ , such that  $\text{Ind}(I) = \text{Ind}(I')$ .*

*Proof.* Set  $I' = F(\text{Ind}(I))$ . Then  $\text{Ind}(I) = \text{Ind}(I')$ , by (4.5.iii). Moreover

$$F(\text{Ind}(I')) = F(\text{Ind}(I)) = I',$$

so  $I'$  is admissible. If  $I'$  and  $I''$  are two admissible ideals inducing the same ideal of  $\mathcal{L}_c(X) \rtimes G$ , then

$$I' = F(\text{Ind}(I')) = F(\text{Ind}(I'')) = I''. \quad \square$$

We have in fact already encountered examples of admissible ideals:

**4.8. Proposition.** *The two trivial ideals of  $KH$ , namely  $\{0\}$  and  $KH$ , itself, are admissible.*

*Proof.* Setting  $I = \{0\}$ , we have that

$$I = \{0\} \subseteq F(\text{Ind}(I)) \stackrel{(4.5.ii)}{\subseteq} I,$$

so  $I$  is admissible.

On the other hand, if  $I = KH$ , we have seen in (3.15) that  $\text{Ind}(I) = \mathcal{L}_c(X) \rtimes G$ , so

$$F(\text{Ind}(I)) = F(\mathcal{L}_c(X) \rtimes G) = KH = I,$$

so, again,  $I$  is seen to be admissible.  $\square$

In order to better understand admissible ideals, we must be able to describe the image of an ideal in  $\mathcal{L}_c(X) \rtimes G$  through  $F$ .

**4.9. Proposition.** *Let  $J \trianglelefteq \mathcal{L}_c(X) \rtimes G$  be any ideal and let  $c = \sum_{h \in H} c_h \delta_h$  be any element of  $KH$ . Then  $c$  is in  $F(J)$  if and only if there exists a compact-open set  $V$ , such that*

$$x_0 \in V \subseteq X_h,$$

whenever  $c_h \neq 0$ , satisfying

$$c_V := \sum_{h \in H} c_h 1_V \Delta_h \in J.$$

*Proof.* Assuming that  $c_V$  is in  $J$ , we have

$$F(J) \ni F(c_V) = \sum_{h \in H} c_h 1_V(x_0) \delta_h = \sum_{h \in H} c_h \delta_h = c,$$

so  $c \in F(J)$ . Conversely, if  $c \in F(J)$ , pick  $b$  in  $J$  such that  $c = F(b)$ . We will initially prove that  $b$  may be chosen in  $\mathcal{L}_c(X) \rtimes H$ .

Write  $b = \sum_{g \in \Gamma} f_g \Delta_g$ , where  $\Gamma$  is a finite subset of  $G$ , and set

$$\Gamma_1 = \{g \in \Gamma : x_0 \notin X_{g^{-1}}\},$$

$$\Gamma_2 = \{g \in \Gamma : x_0 \in X_{g^{-1}}, \theta_g(x_0) \neq x_0\},$$

$$\Gamma_3 = \{g \in \Gamma : x_0 \in X_{g^{-1}}, \theta_g(x_0) = x_0\} = \Gamma \cap H.$$

It is then clear that  $\Gamma$  is the disjoint union of the  $\Gamma_i$ .

For each  $g$  in  $\Gamma_1$ , using that  $X_{g^{-1}}$  is closed, choose a compact-open set  $W_g$ , such that

$$x_0 \in W_g \subseteq X \setminus X_{g^{-1}}.$$

For each  $g$  in  $\Gamma_2$ , choose open sets  $U$  and  $V$ , such that  $x_0 \in U$ ,  $\theta_g(x_0) \in V$ , and  $U \cap V = \emptyset$ . By replacing  $V$  with  $V \cap X_g$  we may assume that  $V \subseteq X_g$ . We then set  $Z = U \cap \theta_{g^{-1}}(V)$ , and observe that  $x_0 \in Z \subseteq X_{g^{-1}}$ , and that

$$Z \cap \theta_g(Z) \subseteq U \cap \theta_g(\theta_{g^{-1}}(V)) = U \cap V = \emptyset.$$

Choosing a compact-open neighborhood  $W_g$  of  $x_0$  contained in  $Z$ , we then have that

$$x_0 \in W_g \subseteq X_{g^{-1}}, \quad \text{and} \quad W_g \cap \theta_g(W_g) = \emptyset.$$

Ignoring  $\Gamma_3$  for the time being we put

$$W = \bigcap_{g \in \Gamma_1 \cup \Gamma_2} W_g,$$

and observe that

$$W \cap X_{g^{-1}} = \emptyset, \quad \forall g \in \Gamma_1, \quad \text{and}$$

$$x_0 \in W \subseteq X_{g^{-1}}, \quad \text{and} \quad W \cap \theta_g(W) = \emptyset, \quad \forall g \in \Gamma_2.$$

We then have that

$$\begin{aligned} 1_W b 1_W &= \sum_{g \in \Gamma} 1_W (f_g \Delta_g) 1_W = \sum_{g \in \Gamma} 1_W f_g \alpha_g(1_W 1_{X_{g^{-1}}}) \Delta_g = \sum_{g \in \Gamma} 1_W f_g \alpha_g(1_{W \cap X_{g^{-1}}}) \Delta_g = \\ &= \sum_{g \in \Gamma} 1_W f_g 1_{\theta_g(W \cap X_{g^{-1}})} \Delta_g = \sum_{g \in \Gamma} f_g 1_{W \cap \theta_g(W \cap X_{g^{-1}})} \Delta_g. \end{aligned}$$

For  $g \in \Gamma_1 \cup \Gamma_2$  we have that  $W \cap \theta_g(W \cap X_{g^{-1}}) = \emptyset$ , so the summand corresponding to  $g$  in the above sum vanishes. Therefore,

$$1_W b 1_W = \sum_{g \in \Gamma_3} f_g 1_{W \cap \theta_g(W \cap X_{g^{-1}})} \Delta_g = \sum_{g \in \Gamma_3} f'_g \Delta_g,$$

where the  $f'_g$  are defined by the last equality above. Setting  $b' = 1_W b 1_W$ , we then have that  $b' \in \mathcal{L}_c(X) \rtimes H$ , because  $\Gamma_3 = \Gamma \cap H$ , and moreover  $b' \in J$ . Recalling that  $c = F(b)$ , we also have that

$$F(b') = F(1_W b 1_W) \stackrel{(4.2)}{=} 1_W(x_0) F(b) 1_W(x_0) = F(b) = c.$$

Replacing  $b$  by  $b'$  we have therefore proven our claim that  $b$  may be chosen in  $\mathcal{L}_c(X) \rtimes H$ , so we are allowed to write

$$b = \sum_{h \in H} f_h \Delta_h.$$

For each  $h$  in  $H$ , choose a compact-open set  $V_h \subseteq X_h$  such that  $x_0 \in V_h$ , and such that  $f_h$  is constant on  $V_h$ . Letting  $V$  be the intersection of the finitely many  $V_h$  for which  $f_h$  is nonzero, we have that the  $f_h$  are constant on  $V$ , so that  $1_V f_h = d_h 1_V$ , where  $d_h$  is the constant value attained by  $f_h$  on  $V$ .

We may then define

$$b'' := 1_V b = \sum_{h \in H} 1_V f_h \Delta_h = \sum_{h \in H} d_h 1_V \Delta_h,$$

observing that, as above,  $b'' \in J$ , and  $F(b'') = c$ . The latter may be expressed as

$$\sum_{h \in H} c_h \delta_h = F\left(\sum_{h \in H} d_h 1_V \Delta_h\right) = \sum_{h \in H} d_h 1_V(x_0) \delta_h = \sum_{h \in H} d_h \delta_h.$$

It follows that  $d_h = c_h$ , for all  $h$ , whence

$$c_V = \sum_{h \in H} c_h 1_V \Delta_h = \sum_{h \in H} d_h 1_V \Delta_h = b'' \in J. \quad \square$$

We may now employ (4.9) to give a characterization of admissible ideals.

**4.10. Proposition.** *An ideal  $I \trianglelefteq KH$  is admissible if and only if, for every  $c = \sum_{h \in H} c_h \delta_h$  in  $I$ , there exists a neighborhood  $V$  of  $x_0$ , such that*

$$\delta_{k^{-1}} \left( \sum_{h \in H \cap kHl^{-1}} c_h \delta_h \right) \delta_l \in I,$$

for all  $k$  and  $l$  in  $\mathcal{S}$ , such that  $\theta_k(x_0) \in V$ .

*Proof.* Supposing that  $I$  is admissible, pick  $c = \sum_{h \in H} c_h \delta_h \in I$ . By hypothesis  $c$  is in  $F(\text{Ind}(I))$ , so (4.9) provides a compact-open set  $V \ni x_0$ , such that

$$c_V := \sum_{h \in H} c_h 1_V \Delta_h \in \text{Ind}(I).$$

In view of the definition of  $\text{Ind}(I)$ , one has that  $\langle \delta_k, c_V \delta_l \rangle \in I$ , for every  $k$  and  $l$  in  $\mathcal{S}$ , and if we use (3.13) under the hypothesis that  $\theta_k(x_0) \in V$ , we deduce that

$$I \ni \langle \delta_k, c_V \delta_l \rangle = \sum_{h \in H \cap kHl^{-1}} c_h 1_V(\theta_k(x_0)) \delta_{k^{-1}hl} = \sum_{h \in H \cap kHl^{-1}} c_h \delta_{k^{-1}hl},$$

proving the condition displayed in the statement.

Conversely, assuming that this condition holds, let us show that  $I$  is admissible, namely that  $I \subseteq F(\text{Ind}(I))$ , since the reverse inclusion is granted by (4.5.ii). For this, given  $c = \sum_{h \in H} c_h \delta_h \in I$ , pick  $V$  as in the statement. By shrinking  $V$  a bit, if necessary, we may assume that  $V$  is compact-open and  $V \subseteq X_h$ , whenever  $c_h \neq 0$ , so that

$$c_V := \sum_{h \in H} c_h 1_V \Delta_h$$

is a legitimate element of  $\mathcal{L}_c(X) \rtimes G$ . We then claim that  $c_V \in \text{Ind}(I)$ . To prove this it is enough to verify that  $\langle \delta_k, c_V \delta_l \rangle \in I$ , for all  $k$  and  $l$  in  $\mathcal{S}$ . Using (3.13) again we have

$$\langle \delta_k, c_V \delta_l \rangle = \sum_{h \in H \cap kHl^{-1}} c_h 1_V(\theta_k(x_0)) \delta_{k^{-1}hl} = [\theta_k(x_0) \in V] \delta_{k^{-1}} \left( \sum_{h \in H \cap kHl^{-1}} c_h \delta_h \right) \delta_l.$$

In case  $\theta_k(x_0) \in V$ , the hypothesis implies that the above belongs to  $I$ , and otherwise  $\langle \delta_k, c_V \delta_l \rangle$  vanishes so it also lies in  $I$ . This shows that  $c_V$  is in  $\text{Ind}(I)$ , so

$$c = F(c_V) \in F(\text{Ind}(I)),$$

concluding the proof.  $\square$

Recall from (4.5.ii) that, for every ideal  $I \trianglelefteq KH$ , one has that  $F(\text{Ind}(I)) \subseteq I$ . One may similarly inquire about the relationship between  $J$  and  $\text{Ind}(F(J))$ . The answer is given in our next:

**4.11. Proposition.**

- (i) *For every ideal  $J \trianglelefteq \mathcal{L}_c(X) \rtimes G$ , one has that  $J \subseteq \text{Ind}(F(J))$ .*
- (ii) *For every ideal  $I \trianglelefteq KH$ , one has that  $\text{Ind}(I)$  is the largest among the ideals  $J \trianglelefteq \mathcal{L}_c(X) \rtimes G$  satisfying  $F(J) \subseteq I$ .*

*Proof.* (i) Given  $b$  in  $J$ , notice that for every  $u$  and  $v$  in  $\mathcal{L}_c(X) \rtimes G$ , one has that  $ubv \in J$ , so

$$F(ubv) \in F(J).$$

We then deduce from (3.22) that  $b$  lies in  $\text{Ind}(F(J))$ . This proves (i).

(ii) As already mentioned, (4.5.ii) gives  $F(\text{Ind}(I)) \subseteq I$ , so  $\text{Ind}(I)$  is indeed among the ideals mentioned above. Next, given any ideal  $J \trianglelefteq \mathcal{L}_c(X) \rtimes G$ , with  $F(J) \subseteq I$ , we have

$$J \stackrel{(i)}{\subseteq} \text{Ind}(F(J)) \subseteq \text{Ind}(I). \quad \square$$

Our main interest is to construct ideals in  $\mathcal{L}_c(X) \rtimes G$  from admissible ideals in  $KH$ , but it is interesting to remark that one may also go the other way:

**4.12. Proposition.** *Let  $J \trianglelefteq \mathcal{L}_c(X) \rtimes G$  be any ideal. Then  $F(J)$  is an admissible ideal of  $KH$ .*

*Proof.* By (4.11.i) we have that  $J \subseteq \text{Ind}(F(J))$ . So, if we apply  $F$  on both sides of this inclusion, we get

$$F(J) \subseteq F(\text{Ind}(F(J))) \stackrel{(4.5.ii)}{\subseteq} F(J),$$

so we see that  $F(\text{Ind}(F(J))) = F(J)$ , which is to say that  $F(J)$  is admissible.  $\square$

## 5. Representations.

As before we adopt our standing assumptions (2.1). In this section we will begin the preparations for proving that any ideal (always meaning two-sided ideal) of  $\mathcal{L}_c(X) \rtimes G$  is the intersection of ideals induced from isotropy subgroups.

Our methods will largely rely on representation theory, so we begin by spelling out a trivial connection between representations and ideals.

**5.1. Proposition.** *Let  $B$  be a  $K$ -algebra possessing local units<sup>7</sup> and let us be given an ideal  $J \trianglelefteq B$ . Then there exists a vector space  $V$ , and a non-degenerate<sup>8</sup> representation*

$$\pi : B \rightarrow L(V),$$

such that  $J = \text{Ker}(\pi)$ .

*Proof.* Let  $V = B/J$ , denote the quotient map by  $q : B \rightarrow V$ , and consider the representation  $\pi : B \rightarrow L(V)$  given by

$$\pi(b)q(v) = q(bv), \quad \forall b, v \in B.$$

It is then obvious that  $J \subseteq \text{Ker}(\pi)$ , but the reverse inclusion may also be verified: in fact, if  $b$  is in  $\text{Ker}(\pi)$ , choose  $e$  in  $B$  such that  $b = be$ , so

$$0 = \pi(b)q(e) = q(be) = q(b),$$

whence  $b$  is in  $J$ . In order to show that  $\pi$  is non-degenerate, pick any  $\xi$  in  $V$ , and write  $\xi = q(b)$ , for some  $b$  in  $B$ . Letting  $e$  be such that  $b = eb$ , we have

$$\xi = q(b) = q(eb) = \pi(e)q(b) \in \pi(B)V. \quad \square$$

To see that the above result applies to our situation, we give the following:

**5.2. Proposition.** *For every  $b$  in  $\mathcal{L}_c(X) \rtimes G$ , there is an idempotent  $e \in \mathcal{L}_c(X)$ , such that  $eb = b = be$ . In particular  $\mathcal{L}_c(X) \rtimes G$  has local units.*

*Proof.* Given  $b$  in  $\mathcal{L}_c(X) \rtimes G$ , write

$$b = \sum_{g \in G\Gamma} f_g \Delta_g,$$

where  $\Gamma$  is a finite subset of  $G$ , and each  $f_g$  lies in  $\mathcal{L}_c(X_g)$ . For each  $g$  in  $\Gamma$ , let  $C_g = \text{supp}(f_g)$ , so that  $C_g$  is a compact-open subset of  $X$ , contained in  $X_g$ . Put

$$C = \bigcup_{g \in \Gamma} (C_g \cup \theta_{g^{-1}}(C_g)).$$

<sup>7</sup> Recall that  $B$  is said to have local units if, for every  $b$  in  $B$ , there exists an idempotent  $e \in B$ , such that  $eb = b = be$ .

<sup>8</sup> We say that  $\pi$  is non-degenerate if  $V = [\pi(B)V]$ , brackets meaning linear span.



It follows that  $C$  is also a compact-open subset of  $X$ , whence its characteristic function  $1_C$  is an idempotent element of  $\mathcal{L}_c(X)$ , hence also of  $\mathcal{L}_c(X) \rtimes G$ .

We next claim that  $1_C b = b 1_C = b$ . To see this we observe that, for obvious reasons,  $1_C f_g = f_g$ , for any  $g$  in  $\Gamma$ , so it is clear that  $1_C b = b$ . On the other hand

$$b 1_C = \sum_{g \in G\Gamma} f_g \Delta_g 1_C \stackrel{(2.3)}{=} \sum_{g \in G\Gamma} f_g \bar{\alpha}_g(1_C) \Delta_g, \quad (5.2.1)$$

while, for every  $g$  in  $\Gamma$ , we have that

$$f_g \bar{\alpha}_g(1_C) = 1_{C_g} f_g \alpha(1_C 1_{g^{-1}}) = 1_{C_g} f_g 1_{\theta_g(C \cap X_{g^{-1}})} = f_g 1_{C_g \cap \theta_g(C \cap X_{g^{-1}})} = f_g 1_{C_g} = f_g,$$

where, in the penultimate step we have used that

$$C_g = \theta_g(\theta_{g^{-1}}(C_g)) = \theta_g(\theta_{g^{-1}}(C_g) \cap X_{g^{-1}}) \subseteq \theta_g(C \cap X_{g^{-1}}).$$

This proves that  $f_g \bar{\alpha}_g(1_C) = f_g$ , whence the computation in (5.2.1) gives that  $b 1_C = b$ .  $\square$

From this point on, we will fix an arbitrary ideal  $J \trianglelefteq \mathcal{L}_c(X) \rtimes G$ , which in view of (5.1) and (5.2), we may assume is the kernel of a likewise fixed non-degenerate representation

$$\pi : \mathcal{L}_c(X) \rtimes G \rightarrow L(V).$$

For the time being we will forget about the ideal  $J$  mentioned above, and we will mostly focus our attention on the representation  $\pi$ , even though our main long term goal is to study  $J$ .

**5.3. Proposition.** *Regarding the above representation  $\pi$ , its restriction to  $\mathcal{L}_c(X)$  is non-degenerate.*

*Proof.* Given any vector  $\xi$  in  $V$ , write

$$\xi = \sum_{i=1}^n \pi(b_i) \xi_i,$$

with  $b_i$  in  $\mathcal{L}_c(X) \rtimes G$ , and  $\xi_i$  in  $V$ . Using (5.2), for each  $i$  we choose an idempotent  $e_i \in \mathcal{L}_c(X)$  such that  $b_i = e_i b_i$ , so

$$\xi = \sum_{i=1}^n \pi(b_i) \xi_i = \sum_{i=1}^n \pi(e_i b_i) \xi_i = \sum_{i=1}^n \pi(e_i) \pi(b_i) \xi_i \in [\pi(\mathcal{L}_c(X))V]. \quad \square$$

**5.4. Proposition.** (Disintegration) *There exists a partial representation*

$$u : G \rightarrow L(V),$$

*such that, for all  $g \in G$ , and  $f \in \mathcal{L}_c(X)$ , one has*

- (i)  $u_g \pi(f) = \pi(\bar{\alpha}_g(f)) u_g$ ,
- (ii)  $\pi(f \Delta_g) = \pi(f) u_g$ , provided  $f \in \mathcal{L}_c(X_g)$ ,
- (iii)  $u_g u_{g^{-1}} \pi(f) = \pi(f 1_g) = \pi(f) u_g u_{g^{-1}}$ .

*Proof.* Given any  $\xi$  in  $V$ , use (5.3) to write

$$\xi = \sum_{i=1}^n \pi(\varphi_i) \eta_i,$$

where each  $\varphi_i \in \mathcal{L}_c(X)$ , and  $\eta_i \in V$ . We then define

$$u_g \xi = \sum_{i=1}^n \pi(\bar{\alpha}_g(\varphi_i) \Delta_g) \eta_i. \quad (5.4.1)$$

To prove that this is well defined, suppose that  $\xi = 0$ , and let

$$V = \bigcup_{i=1}^n \text{supp}(\varphi_i) \cap X_{g^{-1}}.$$

So  $V$  is a compact-open set and  $1_V \varphi_i 1_{g^{-1}} = \varphi_i 1_{g^{-1}}$ , for all  $i = 1, \dots, n$ . Therefore

$$1_{\theta_g(V)} \Delta_g \varphi_i = \alpha_g(1_V) \bar{\alpha}_g(\varphi_i) \Delta_g = \bar{\alpha}_g(1_V \varphi_i) \Delta_g = \bar{\alpha}_g(\varphi_i) \Delta_g,$$

so the right-hand-side of (5.4.1) coincides with

$$\sum_{i=1}^n \pi(1_{\theta_g(V)} \Delta_g \varphi_i) \eta_i = \pi(1_{\theta_g(V)} \Delta_g) \sum_{i=1}^n \pi(\varphi_i) \eta_i = \pi(1_{\theta_g(V)} \Delta_g) \xi = 0.$$

This shows that  $u_g$  is well defined.

In order to prove (i), consider a vector  $\xi \in V$  of the form  $\xi = \pi(\varphi)\eta$ , for some  $\varphi \in \mathcal{L}_c(X)$ , and  $\xi\eta \in V$ , and observe that

$$\begin{aligned} u_g \pi(f) \xi &= u_g \pi(f) \pi(\varphi) \eta = u_g \pi(f\varphi) \eta = \pi(\bar{\alpha}_g(f\varphi) \Delta_g) \eta = \\ &= \pi(\bar{\alpha}_g(f)) \pi(\bar{\alpha}_g(\varphi) \Delta_g) \eta = \pi(\bar{\alpha}_g(f)) u_g \xi. \end{aligned}$$

Since the set of vectors  $\xi$  of the above form spans  $V$ , we have proved (i).

With the goal of proving (ii), let  $\xi = \pi(\varphi)\eta$ , as above, and notice that

$$\begin{aligned} \pi(f \Delta_g) \xi &= \pi(f \Delta_g) \pi(\varphi) \eta = \pi(f \Delta_g \varphi) \eta = \\ &= \pi(f \bar{\alpha}_g(\varphi) \Delta_g) \eta = \pi(f) \pi(\bar{\alpha}_g(\varphi) \Delta_g) \eta = \pi(f) u_g \xi. \end{aligned}$$

In order to prove (iii) write  $f = f_1 f_2$ , with  $f_1, f_2 \in \mathcal{L}_c(X)$ , and let  $\xi \in V$ . Then

$$\begin{aligned} u_g u_{g^{-1}} \pi(f) \xi &= u_g \pi(\bar{\alpha}_{g^{-1}}(f) \Delta_{g^{-1}}) \xi = u_g \pi(\bar{\alpha}_{g^{-1}}(f_1)) \pi(\bar{\alpha}_{g^{-1}}(f_2) \Delta_{g^{-1}}) \xi = \\ &= \pi(\bar{\alpha}_g(\bar{\alpha}_{g^{-1}}(f_1)) \Delta_g) \pi(\bar{\alpha}_{g^{-1}}(f_2) \Delta_{g^{-1}}) \xi = \pi(f_1 1_g \bar{\alpha}_{g^{-1}}(f_2) \Delta_{g^{-1}}) \xi = \\ &= \pi(f_1 1_g \bar{\alpha}_g(\bar{\alpha}_{g^{-1}}(f_2))) \xi = \pi(f_1 1_g f_2 1_g) \xi = \pi(f 1_g) \xi. \end{aligned}$$

This proves the first identity in (iii). As for the second, let  $\xi = \pi(\varphi)\eta$ , so

$$\pi(f) u_g u_{g^{-1}} \xi = \pi(f) u_g u_{g^{-1}} \pi(\varphi) \eta = \pi(f) \pi(\varphi 1_g) \eta = \pi(f \varphi 1_g) \eta = \pi(f 1_g) \pi(\varphi) \eta = \pi(f 1_g) \xi.$$

This concludes the proof of (iii).

We leave it for the reader to prove that  $u$  is a partial representation.  $\square$

We are now about to take a major step in our quest to understand general ideals in terms of induced ones. Observe that the very definition of induced ideals requires that a point of  $X$  be chosen in advance, so we must begin to see our representation  $\pi$  from the point of view of a chosen point in  $X$ , a process which will eventually lead to a *discretization* of  $\pi$  (see (5.4) below). This will be accomplished by means of the following device: for each  $x$  in  $X$ , let

$$I_x = \{f \in \mathcal{L}_c(X) : f(x) = 0\},$$

which is clearly an ideal in  $\mathcal{L}_c(X)$ . Consequently

$$Z_x := [\pi(I_x)V]$$

is invariant under  $\mathcal{L}_c(X)$ , so there is a well defined representation  $\pi_x$  of  $\mathcal{L}_c(X)$  on

$$V_x := V/Z_x,$$

making the following diagram commute

$$\begin{array}{ccc} V & \xrightarrow{\pi(f)} & V \\ q_x \downarrow & & \downarrow q_x \\ V_x & \xrightarrow{\pi_x(f)} & V_x \end{array}$$

The first indication that our localization process is bearing fruit is as follows:

**5.5. Proposition.** *Given any  $x$  in  $X$ , and any  $f$  in  $\mathcal{L}_c(X)$ , one has that*

$$\pi_x(f)\eta = f(x)\eta,$$

for every  $\eta$  in  $V_x$ .

*Proof.* Using (5.3) it is enough to verify the statement for  $\eta$  of the form  $\eta = q_x(\pi(\varphi)\xi)$ , where  $\varphi \in \mathcal{L}_c(X)$ , and  $\xi \in V$ . Choose a compact-open set  $C$  containing  $\text{supp}(\varphi) \cup \{x\}$ , and observe that  $1_C\varphi = \varphi$ . In addition  $f - f(x)1_C$  lies in  $I_x$ , since it clearly vanishes at  $x$ . Therefore

$$\pi_x(f)\eta = \pi_x(f)q_x(\pi(\varphi)\xi) = q_x(\pi(f)\pi(\varphi)\xi) = q_x(\pi(f\varphi)\xi) = \dots \quad (5.5.1)$$

Notice that

$$f\varphi = (f(x)1_C + f - f(x)1_C)\varphi = f(x)1_C\varphi + (f - f(x)1_C)\varphi \stackrel{(\text{mod } I_x)}{\equiv} f(x)\varphi,$$

so

$$\pi(f\varphi)\xi \stackrel{(\text{mod } Z_x)}{\equiv} \pi(f(x)\varphi)\xi = f(x)\pi(\varphi)\xi,$$

and we then conclude that (5.5.1) equals

$$\dots = q_x(f(x)\pi(\varphi)\xi) = f(x)q_x(\pi(\varphi)\xi) = f(x)\eta. \quad \square$$

Putting together the definition of  $\pi_x$  with the result above, we get the following useful formulas:

$$q_x(\pi(f)\xi) = \pi_x(f)q_x(\xi) = f(x)q_x(\xi), \quad (5.6)$$

for all  $x \in X$ ,  $f \in \mathcal{L}_c(X)$ , and  $\xi \in V$ .

Having focused on  $\mathcal{L}_c(X)$ , we momentarily lost track of the  $u_g$ , but there is still time to bring them back into focus:

**5.7. Proposition.** *If  $x$  is in  $X_{g^{-1}}$ , then:*

- (i)  $u_g(Z_x) \subseteq Z_{\theta_g(x)}$ , where  $u$  is as in (5.4),
- (ii) *there exists a linear mapping*

$$\mu_g^x : V_x \rightarrow V_{\theta_g(x)},$$

such that

$$\mu_g^x(q_x(\xi)) = q_{\theta_g(x)}(u_g\xi), \quad \forall \xi \in V.$$

*Proof.* (i) Let  $\xi$  be a vector in  $Z_x$  of the form  $\xi = \pi(\varphi)\eta$ , where  $\varphi \in I_x$ , and  $\eta \in V$ . Then

$$u_g\xi = u_g\pi(\varphi)\eta \stackrel{(5.4.i)}{=} \pi(\bar{\alpha}_g(\varphi))u_g\eta.$$

Notice that  $\bar{\alpha}_g(\varphi)$  lies in  $I_{\theta_g(x)}$ , because

$$\bar{\alpha}_g(\varphi)|_{\theta_g(x)} = \varphi(\theta_{g^{-1}}(\theta_g(x))) = \varphi(x) = 0,$$

whence  $u_g\xi \in Z_{\theta_g(x)}$ .

(ii) Follows immediately from (i).  $\square$

The  $\mu_g^x$  obey the following functorial property:

**5.8. Proposition.** *If  $x \in X_{g^{-1}} \cap X_{g^{-1}h^{-1}}$ , then the composition*

$$V_x \xrightarrow{\mu_g^x} V_{\theta_g(x)} \xrightarrow{\mu_h^{\theta_g(x)}} V_{\theta_{hg}(x)}$$

coincides with  $\mu_{hg}^x$ .

*Proof.* We initially claim that for all  $g$  in  $G$ , if  $x \in X_g$ , then

$$q_x(\xi) = q_x(u_g u_{g^{-1}} \xi), \quad \forall \xi \in V.$$

This will clearly follow should we prove that

$$\xi - u_g u_{g^{-1}} \xi \in Z_x, \quad \forall \xi \in V,$$

which we will now do. By (5.3), we may assume that  $\xi = \pi(\varphi)\eta$ , for some  $\varphi$  in  $\mathcal{L}_c(X)$ , and  $\eta$  in  $V$ . We then have

$$\xi - u_g u_{g^{-1}} \xi = \pi(\varphi)\eta - u_g u_{g^{-1}} \pi(\varphi)\eta \stackrel{(5.4.iii)}{=} \pi(\varphi)\eta - \pi(\varphi 1_g)\eta = \pi(\varphi - \varphi 1_g)\eta.$$

Observing that  $\varphi - \varphi 1_g$  is in  $I_x$ , we have that  $\pi(\varphi - \varphi 1_g)\eta$  lies in  $Z_x$ , proving the claim.

Addressing the statement, choose any element of  $V_x$ , say  $q_x(\xi)$ , for some  $\xi$  in  $V$ , and notice that

$$\begin{aligned} \mu_h^{\theta_g(x)}(\mu_g^x(q_x(\xi))) &= \mu_h^{\theta_g(x)}(q_{\theta_g(x)}(u_g \xi)) = q_{\theta_h(\theta_g(x))}(u_h u_g \xi) = \\ &= q_{\theta_{hg}(x)}(u_h u_{h^{-1}} u_h u_g \xi) = q_{\theta_{hg}(x)}(u_h u_{h^{-1}} u_{hg} \xi) = \dots \end{aligned}$$

Since  $\theta_{hg}(x) = \theta_h(\theta_g(x)) \in X_h$ , and thanks to our claim, the above equals

$$\dots \stackrel{(5.8)}{=} q_{\theta_{hg}(x)}(u_{hg} \xi) = \mu_{hg}^x(q_x \xi). \quad \square$$

Let us now consider the representation

$$\Pi = \prod_{x \in X} \pi_x$$

of  $\mathcal{L}_c(X)$  on the cartesian product  $\prod_{x \in X} V_x$ . Thus, if  $f \in \mathcal{L}_c(X)$ , and  $\eta = (\eta_x)_{x \in X} \in \prod_{x \in X} V_x$ , we have

$$(\Pi(f)\eta)_x = \pi_x(f)\eta_x, \quad \forall x \in X.$$

Incidentally, by (5.5) the term  $\pi_x(f)\eta_x$ , above, could be replaced by  $f(x)\eta_x$ , if desired. Thus,  $\Pi(f)$  is the block diagonal operator, acting on each  $V_x$  as the scalar multiplication by  $f(x)$ .

Also, for each  $g$  in  $G$ , consider the linear operator  $U_g$  on  $\prod_{x \in X} V_x$ , given by

$$U_g(\eta)_x = [x \in X_g] \mu_g(\eta_{\theta_{g^{-1}}(x)}), \quad \forall \eta = (\eta_x)_{x \in X} \in \prod_{x \in X} V_x. \quad (5.9)$$

The above occurrence of  $\mu_g$  should have actually been written as  $\mu_g^{\theta_{g^{-1}}(x)}$ , but due to the awkward nature of this notation we will rely on the context to determine the missing superscript.

In what amounts to be essentially a rewording of (5.8), we have:

**5.10. Proposition.** *Identifying  $V_x$  as a subspace of  $\prod_{x \in X} V_x$ , in the natural way, we have:*

- (i) *if  $x \notin X_{g^{-1}}$ , then  $U_g$  vanishes on  $V_x$ ,*
- (ii) *if  $x \in X_{g^{-1}}$ , then  $U_g$  coincides with  $\mu_g^x$ , and hence maps  $V_x$  into  $V_{\theta_g(x)}$ ,*
- (iii) *if  $x \in X_{g^{-1}}$ , then  $U_g$  maps  $V_x$  bijectively onto  $V_{\theta_g(x)}$ ,*
- (iv) *if  $x \in X_{g^{-1}} \cap X_{g^{-1}h^{-1}}$ , then the composition*

$$V_x \xrightarrow{U_g} V_{\theta_g(x)} \xrightarrow{U_h} V_{\theta_{hg}(x)}$$

*coincides with  $U_{hg}$  on  $V_x$ .*

*Proof.* (i) and (ii) follow easily by inspection, while (iv) follows directly from (5.8).

In order to prove that  $U_g$  is bijective from  $V_x$  to  $V_{\theta_g(x)}$ , it is enough to observe that, by (iv), the restriction of  $U_{g^{-1}}$  to  $V_{\theta_g(x)}$  is the inverse of  $U_g$ .  $\square$

**5.11. Proposition.** *For every  $g$  in  $G$ , and every  $f \in \mathcal{L}_c(X)$ , one has that*

$$U_g \Pi(f) = \Pi(\bar{\alpha}_g(f)) U_g.$$

*Proof.* Given  $\eta = (\eta_x)_{x \in X} \in V$ , one has for every  $x$  in  $X$  that

$$\begin{aligned} (U_g \Pi(f) \eta)_x &= [x \in X_g] \mu_g \left( (\Pi(f) \eta)_{\theta_{g^{-1}}(x)} \right) = [x \in X_g] \mu_g \left( f(\theta_{g^{-1}}(x)) \eta_{\theta_{g^{-1}}(x)} \right) = \\ &= [x \in X_g] f(\theta_{g^{-1}}(x)) \mu_g(\eta_{\theta_{g^{-1}}(x)}) = \bar{\alpha}_g(f)|_x (U_g \eta)_x = \left( \Pi(\bar{\alpha}_g(f)) U_g \eta \right)_x. \end{aligned}$$

This concludes the proof.  $\square$

As a consequence, there exists a representation  $\Pi \times U$  of  $\mathcal{L}_c(X) \rtimes G$  on  $\prod_{x \in X} V_x$ , such that

$$(\Pi \times U)(f \Delta_g) = \Pi(f) U_g, \quad \forall f \in \mathcal{L}_c(X_{g^{-1}}).$$

**5.12. Definition.** The representation  $\Pi \times U$  above will be referred to as the *discretization* of the initially given representation  $\pi$ .

**5.13. Proposition.** *The mapping*

$$Q : \xi \in V \mapsto (q_x(\xi))_{x \in X} \in \prod_{x \in X} V_x,$$

*is injective and covariant relative to the corresponding representations of  $\mathcal{L}_c(X) \rtimes G$  on  $V$  and on  $\prod_{x \in X} V_x$ , respectively.*

*Proof.* Let  $g \in G$ , and  $f \in \mathcal{L}_c(X_g)$ . Then, for every  $\xi$  in  $V$ , and every  $x \in X$ , we have

$$\begin{aligned} ((\Pi \times U)(f \Delta_g) Q(\xi))_x &= (\Pi(f) U_g Q(\xi))_x = f(x) (U_g Q(\xi))_x = f(x) [x \in X_g] \mu_g(Q(\xi)_{\theta_{g^{-1}}(x)}) = \\ &= f(x) \mu_g(q_{\theta_{g^{-1}}(x)}(\xi)) = f(x) q_x(u_g \xi) = q_x(\pi(f) u_g \xi) = Q(\pi(f \Delta_g) \xi)_x. \end{aligned}$$

This proves that  $Q$  is covariant.

In order to prove that  $Q$  is injective, suppose that  $Q(\xi) = 0$ , for a given  $\xi$  in  $V$ . We then claim that, for every  $x$  in  $X$ , there exists a compact-open neighborhood  $C$  of  $x$ , such that

$$\pi(1_C) \xi = 0. \tag{5.13.1}$$

To see this, fixing  $x$  in  $X$ , recall that  $q_x(\xi) = 0$ , by hypothesis, so  $\xi$  lies in  $Z_x$  and hence we may write

$$\xi = \sum_{i=1}^n \pi(f_i) \xi_i,$$

where the  $f_i$  are in  $I_x$ , and hence vanish on  $x$ . From the fact that the  $f_i$  are locally constant, and finitely many, it follows that there exists a compact-open neighborhood  $C_x$  of  $x$ , where all of the  $f_i$  vanish. Consequently  $1_{C_x} f_i = 0$ , so

$$\pi(1_{C_x}) \xi = \sum_{i=1}^n \pi(1_{C_x} f_i) \xi_i = 0,$$

proving the claim.

Using (5.3), or recycling any one of the above decompositions of  $\xi$ , let us again write

$$\xi = \sum_{i=1}^n \pi(f_i) \xi_i,$$

with  $f_i \in \mathcal{L}_c(X)$ , and  $\xi_i \in V$ . Let

$$D = \bigcup_{i=1}^n \text{supp}(f_i),$$

so  $D$  is a compact-open subset of  $X$ , and we have

$$\xi = \sum_{i=1}^n \pi(1_D f_i) \xi_i = \pi(1_D) \sum_{i=1}^n \pi(f_i) \xi_i = \pi(1_D) \xi. \quad (5.13.2)$$

Regarding the open cover  $\{C_x\}_{x \in X}$  of  $D$ , where the  $C_x$  are as in the first part of this proof, we may find a finite set  $\{x_1, \dots, x_p\} \subseteq X$ , such that  $D \subseteq \bigcup_{i=1}^p C_{x_i}$ . Putting

$$E_k = D \cap C_{x_k} \setminus \bigcup_{i=1}^{k-1} C_{x_i}, \quad \forall k = 1, \dots, p,$$

it is easy to see that the  $E_k$  are pairwise disjoint compact-open sets, whose union coincides with  $D$ . Observing that  $E_k \subseteq C_{x_k}$ , we then have

$$\xi \stackrel{(5.13.2)}{=} \pi(1_D) \xi = \sum_{k=1}^p \pi(1_{E_k}) \xi = \sum_{k=1}^p \pi(1_{E_k} 1_{C_{x_k}}) \xi = \sum_{k=1}^p \pi(1_{E_k}) \pi(1_{C_{x_k}}) \xi \stackrel{(5.13.1)}{=} 0.$$

This proves that  $Q$  is injective.  $\square$

As an immediate consequence we have

**5.14. Corollary.** *The null space of  $\Pi \times U$  is contained in the null space of  $\pi$ .*

*Proof.* By (5.13) we see that  $\pi$  is equivalent to a subrepresentation of  $\Pi \times U$ , so the conclusion follows.  $\square$

From now on we will consider the subspace

$$\bigoplus_{x \in X} V_x \subseteq \prod_{x \in X} V_x,$$

consisting of the vectors with finitely many nonzero coordinates. It is easy to see that this subspace is invariant under  $\Pi(f)$ , for all  $f$  in  $\mathcal{L}_c(X)$ , as well as under  $U_g$ , for all  $g$  in  $G$ , consequently it is also invariant under  $\Pi \times U$ .

**5.15. Proposition.** *The null space of the representation obtained by restricting  $\Pi \times U$  to  $\bigoplus_{x \in X} V_x$  coincides with the null space of  $\Pi \times U$  itself.*

*Proof.* Given  $b \in \mathcal{L}_c(X) \rtimes G$ , we must show that if  $(\Pi \times U)(b)$  vanishes on  $\bigoplus_{x \in X} V_x$ , then it vanishes everywhere. Writing

$$b = \sum_{g \in G} f_g \Delta_g,$$

we have for every  $\eta = (\eta_x)_{x \in X}$  in  $\prod_{x \in X} V_x$ , and for every  $x \in X$ , that

$$((\Pi \times U)(b)\eta)_x = \sum_{g \in G} (\Pi(f_g) U_g \eta)_x = \sum_{g \in G} f_g(x) [x \in X_g] \mu_g(\eta_{\theta_{g^{-1}}(x)}).$$

From this we see that the  $x^{\text{th}}$  coordinate of  $(\Pi \times U)(b)\eta$  depends only on the coordinates  $\eta_y$ , for  $y$  of the form  $y = \theta_{g^{-1}}(x)$ , where  $g$  is such that  $f_g \neq 0$ , and  $x \in X_g$ . What matters to us is that the set of  $y$ 's mentioned above is finite, so if  $\eta'$  is defined to have the same  $y$ -coordinates as  $\eta$ , for  $y$  on the above finite set, and zero elsewhere, then  $\eta'$  lies in  $\bigoplus_{x \in X} V_x$ , and

$$((\Pi \times U)(b)\eta)_x = ((\Pi \times U)(b)\eta')_x = 0.$$

Since  $\eta$  and  $x$  are arbitrary we deduce that  $(\Pi \times U)(b) = 0$ , concluding the proof.  $\square$

As we turn our attention to the restriction of  $\Pi \times U$  to  $\bigoplus_{x \in X} V_x$ , it is useful to analyze some of its important aspects. Initially, regarding the space where it acts, we will identify each  $V_x$  as a subspace of  $\bigoplus_{x \in X} V_x$ , in the usual way. Thus, given  $\xi$  in  $V$ , we will think of  $q_x(\xi)$  as the element of  $\bigoplus_{x \in X} V_x$  whose coordinates all vanish, except for the  $x^{\text{th}}$  coordinate which takes on the value  $q_x(\xi)$ . Once this is agreed upon, one may easily show that

$$\begin{aligned} \Pi(f)q_x(\xi) &= \pi_x(f)q_x(\xi) = q_x(\pi(f)\xi), \\ U_g(q_x(\xi)) &= [x \in X_{g^{-1}}] \mu_g^x(q_x(\xi)) = [x \in X_{g^{-1}}] q_{\theta_g(x)}(u_g \xi), \end{aligned} \quad (5.16)$$

for all  $f \in \mathcal{L}_c(X)$ ,  $g \in G$ ,  $x \in X$ , and  $\xi \in V$ .

Since  $\bigoplus_{x \in X} V_x$  is spanned by the union of the  $V_x$ , each of which is the range of the corresponding  $q_x$ , the formulas above determine the action of the  $\Pi(f)$  and of the  $U_g$  on the whole space  $\bigoplus_{x \in X} V_x$ . Putting them together, we may give the following concrete description of the restriction of  $\Pi \times U$  to  $\bigoplus_{x \in X} V_x$ :

**5.17. Proposition.** *Given  $b = \sum_{g \in G} f_g \Delta_g$  in  $\mathcal{L}_c(X) \rtimes G$ , one has for all  $x$  in  $X$  and  $\xi$  in  $V$ , that*

$$(\Pi \times U)(b)q_x(\xi) = \sum_{g \in G} [x \in X_{g^{-1}}] q_{\theta_g(x)}(\pi(f_g)u_g \xi).$$

*Proof.* The proof is now a simple direct computation:

$$\begin{aligned} (\Pi \times U)(b)q_x(\xi) &= \sum_{g \in G} \Pi(f_g)U_g(q_x(\xi)) = \sum_{g \in G} \Pi(f_g) [x \in X_{g^{-1}}] q_{\theta_g(x)}(u_g \xi) = \\ &= \sum_{g \in G} [x \in X_{g^{-1}}] q_{\theta_g(x)}(\pi(f_g)u_g \xi). \end{aligned} \quad \square$$

Let us now use this to describe the *matrix entries* of the operator  $(\Pi \times U)(b)$  acting on  $\bigoplus_{x \in X} V_x$ . By this we mean that, for each  $x$  and  $y$  in  $X$ , we want an expression for the  $y^{\text{th}}$  component of the vector obtained by applying  $(\Pi \times U)(b)$  to any given vector in  $V_x$ , say of the form  $q_x(\xi)$ , where  $\xi \in V$ .

The answer is of course the  $y^{\text{th}}$  component of the expression given in (5.17), which is in turn given by the partial sum corresponding to the terms for which  $\theta_g(x) = y$ . The desired expression for matrix entries therefore becomes

$$((\Pi \times U)(b)q_x(\xi))_y = \sum_{\substack{g \in G \\ \theta_g(x)=y}} q_{\theta_g(x)}(\pi(f_g)u_g \xi) = q_y \left( \sum_{\substack{g \in G \\ \theta_g(x)=y}} \pi(f_g)u_g \xi \right). \quad (5.18)$$

Recall that in (5.14) and (5.15) we proved the following relations among the null spaces of  $\pi$ ,  $\Pi \times U$ , and the restriction of the latter to  $\bigoplus_{x \in X} V_x$ :

$$\text{Ker}(\pi) \supseteq \text{Ker}(\Pi \times U) = \text{Ker}(\Pi \times U|_{\bigoplus_{x \in X} V_x}). \quad (5.19)$$

We will now show that equality in fact holds throughout.

**5.20. Theorem.** *The null space of the representation obtained by restricting  $\Pi \times U$  to  $\bigoplus_{x \in X} V_x$  coincides with the null space of  $\pi$ .*

*Proof.* An important aspect of (5.18), to be used shortly, is that since  $(\Pi \times U)(b)$  is well defined on each  $V_x$ , then so is the right-hand-side in (5.18). Precisely speaking, if  $\xi$  and  $\xi'$  are elements of  $V$  such that  $q_x(\xi) = q_x(\xi')$ , then

$$q_y \left( \sum_{\substack{g \in G \\ \theta_g(x)=y}} \pi(f_g)u_g \xi \right) = q_y \left( \sum_{\substack{g \in G \\ \theta_g(x)=y}} \pi(f_g)u_g \xi' \right). \quad (5.20.1)$$

By (5.19), in order to prove the statement, it suffices to prove that if  $b$  is in the null space of  $\pi$ , then  $(\Pi \times U)(b)$  vanishes on  $\bigoplus_{x \in X} V_x$ , which is the same as saying that its matrix entries given by (5.18) vanish for all  $x$  and  $y$  in  $X$ .

Again writing  $b = \sum_{g \in G} f_g \Delta_g$ , let  $\Gamma$  be the subset of  $G$  consisting of those  $g$  for which  $f_g \neq 0$ , and notice that  $\Gamma$  decomposes as the disjoint union of the following subsets:

$$\begin{aligned}\Gamma_1 &= \{g \in \Gamma : y \notin X_g\}, \\ \Gamma_2 &= \{g \in \Gamma : y \in X_g, \theta_{g^{-1}}(y) \neq x\}, \\ \Gamma_3 &= \{g \in \Gamma : y \in X_g, \theta_{g^{-1}}(y) = x\}.\end{aligned}$$

From our hypothesis that  $\pi(b) = 0$ , we conclude that, for every  $\eta$  in  $V$ , one has

$$0 = \pi(b)\eta = \sum_{g \in \Gamma} \pi(f_g \Delta_g)\eta = \sum_{g \in \Gamma} \pi(f_g)u_g\eta. \quad (5.20.2)$$

This looks enticingly like the last part of (5.18), except of course that here we are summing over all of  $\Gamma$ , while only the terms corresponding to  $\Gamma_3$  are being considered there. In order to fix this discrepancy, notice that  $x$  is not a member of the finite set  $\{\theta_{g^{-1}}(y) : g \in \Gamma_2\}$ , so we may choose some  $\varphi$  in  $\mathcal{L}_c(X)$  such that  $\varphi(x) = 1$ , and  $\varphi(\theta_{g^{-1}}(y)) = 0$ , for all  $g \in \Gamma_2$ . Observing that

$$q_x(\pi(\varphi)\xi) \stackrel{(5.6)}{=} \varphi(x)q_x(\xi) = q_x(\xi),$$

we will later use (5.20.1) in order to replace  $\xi$  by

$$\xi' := \pi(\varphi)\xi$$

in (5.18). Meanwhile we claim that

$$q_y(\pi(f_g)u_g\xi') = 0, \quad \forall g \in \Gamma_1 \cup \Gamma_2. \quad (5.20.3)$$

In order to prove this, observe that

$$q_y(\pi(f_g)u_g\xi') = q_y(\pi(f_g)u_g\pi(\varphi)\xi) = q_y(\pi(f_g\bar{\alpha}_g(\varphi))u_g\xi) \stackrel{(5.6)}{=} f_g(y)\bar{\alpha}_g(\varphi)|_y q_y(u_g\xi).$$

If  $g \in \Gamma_1$ , then the fact that  $f_g$  is supported on  $X_g$  implies that  $f_g(y) = 0$ , so the above expression vanishes. On the other hand, if  $g \in \Gamma_2$ , then

$$\bar{\alpha}_g(\varphi)|_y = \varphi(\theta_{g^{-1}}(y)) = 0,$$

so the above expression again vanishes, and (5.20.3) is proved. Combining this with (5.20.2) we then have

$$\begin{aligned}0 &= q_y\left(\sum_{g \in \Gamma} \pi(f_g)u_g\xi'\right) = q_y\left(\sum_{g \in \Gamma_1} \pi(f_g)u_g\xi'\right) + q_y\left(\sum_{g \in \Gamma_2} \pi(f_g)u_g\xi'\right) + q_y\left(\sum_{g \in \Gamma_3} \pi(f_g)u_g\xi'\right) = \\ &= q_y\left(\sum_{g \in \Gamma_3} \pi(f_g)u_g\xi'\right) \stackrel{(5.20.1)}{=} q_y\left(\sum_{g \in \Gamma_3} \pi(f_g)u_g\xi\right) \stackrel{(5.18)}{=} ((\Pi \times U)(b)q_x(\xi))_y.\end{aligned}$$

This shows that  $(\Pi \times U)(b)$  vanishes on  $\bigoplus_{x \in X} V_x$ , and hence the proof is concluded.  $\square$

This result will have important consequences for our study of ideals in  $\mathcal{L}_c(X) \rtimes G$ . The method we shall adopt will be to start with any ideal  $J \trianglelefteq \mathcal{L}_c(X) \rtimes G$ , and then use (5.1) and (5.2) to find a representation  $\pi$ , as above, such that  $\text{Ker}(\pi) = J$ . By (5.20) we may replace  $\pi$  by  $\Pi \times U$  acting on  $\bigoplus_{x \in X} V_x$ , without affecting null spaces, and it will turn out that the latter is easy enough to understand since it decomposes as a direct sum of very straightforward sub-representations, which we will now describe.



**5.21. Proposition.** *Given any  $x_0$  in  $X$ , one has that*

$$\bigoplus_{x \in \text{Orb}(x_0)} V_x$$

*is invariant under  $\Pi \times U$ .*

*Proof.* By (5.10.ii), this space is invariant under every  $U_g$ . It is also invariant under every  $\Pi(f)$ , since in fact each  $V_x$  has this property. Invariance under  $\Pi \times U$  then follows.  $\square$

We shall now study the representation obtained by restricting  $\Pi \times U$  to the invariant space mentioned above, so we better give it a name:

**5.22. Definition.** Given  $x_0$  in  $X$ , we shall denote the invariant subspace referred to in (5.21) by  $W_{x_0}$ , while the representation of  $\mathcal{L}_c(X) \rtimes G$  obtained by restricting  $\Pi \times U$  to  $W_{x_0}$  will be denoted by  $\rho_{x_0}$ .

If  $R \subseteq X$  is a system of representatives for the orbit relation in  $X$ , namely if  $R$  contains exactly one point of each orbit relative to the action of  $G$  on  $X$ , then surely one has

$$\bigoplus_{x \in X} V_x = \bigoplus_{x_0 \in R} W_{x_0},$$

while the restriction of  $\Pi \times U$  to  $\bigoplus_{x \in X} V_x$  is equivalent to  $\bigoplus_{x_0 \in R} \rho_{x_0}$ .

Before we state the main result of this section we should recall that right after the proof of (5.2) we fixed an arbitrary ideal  $J \trianglelefteq \mathcal{L}_c(X) \rtimes G$ , which incidentally has been forgotten ever since.

**5.23. Theorem.** *Let  $J$  be an arbitrary ideal of  $\mathcal{L}_c(X) \rtimes G$ , and let  $\pi$  be a non-degenerate representation of  $\mathcal{L}_c(X) \rtimes G$ , such that  $J = \text{Ker}(\pi)$ . Considering the representations  $\rho_x$  constructed above, we have*

$$J = \bigcap_{x \in R} \text{Ker}(\rho_x),$$

where  $R \subseteq X$  is any system of representatives for the orbit relation in  $X$ .

*Proof.* The null space of  $\pi$  coincides with the null space of the restriction of  $\Pi \times U$  to  $\bigoplus_{x \in X} V_x$  by (5.20). Since the latter representation is equivalent to the direct sum of the  $\rho_x$ , as seen above, the conclusion is evident.  $\square$

## 6. The representations $\rho_{x_0}$ .

In this section we shall keep the setup of the previous section, such as the ingredients listed in (2.1), the ideal  $J \trianglelefteq \mathcal{L}_c(X) \rtimes G$ , and the representation  $\pi : \mathcal{L}_c(X) \rtimes G \rightarrow L(V)$  fixed there.

The usefulness of Theorem (5.23) in describing  $J$  is obviously proportional to the extent to which we may describe the ideals  $\text{Ker}(\rho_{x_0})$  mentioned there, and the good news is that the representations  $\rho_{x_0}$  are well known to us. In fact they are induced from representations of isotropy group algebras. The main goal of this section is to prove that this is indeed the case.

Our next result refers to the behaviour of the operators

$$U_g : \bigoplus_{x \in X} V_x \rightarrow \bigoplus_{x \in X} V_x,$$

when  $g$  lies in an isotropy group.

**6.1. Proposition.** *Fixing  $x_0$  in  $X$ , let  $H$  be the isotropy group of  $x_0$ . Then, for each  $h$  in  $H$ , one has that  $V_{x_0}$  is invariant under  $U_h$ . Moreover, the restriction of  $U_h$  to  $V_{x_0}$  is an invertible operator and the correspondence*

$$h \in H \mapsto U_h|_{V_{x_0}} \in GL(V_{x_0})$$

*is a group representation.*

*Proof.* Follows immediately from (5.10).  $\square$

The representation of  $H$  on  $V_{x_0}$  referred to in the above Proposition may be integrated to a representation of  $KH$ , which in turn makes  $V_{x_0}$  into a left  $KH$ -module. Applying the machinery of Section (3), we may then form the induced module  $M \otimes V_{x_0}$ , as in (3.6), which we may also view as a representation of  $\mathcal{L}_c(X) \rtimes G$  on  $M \otimes V_{x_0}$ .

**6.2. Theorem.** *For each  $x_0$  in  $X$ , one has that  $\rho_{x_0}$  is equivalent to the representation induced from the left  $KH$ -module  $V_{x_0}$ , as described above.*

*Proof.* Recalling from (5.22) that the space of  $\rho_{x_0}$  is

$$W_{x_0} = \bigoplus_{x \in \text{Orb}(x_0)} V_x,$$

consider the bilinear map  $T : M \times V_{x_0} \rightarrow W_{x_0}$  given by

$$T\left(\sum_{k \in \mathcal{S}} c_k \delta_k, \xi\right) = \sum_{k \in \mathcal{S}} c_k U_k(\xi).$$

Recalling that  $M$  is a right  $KH$ -module, and viewing  $V_{x_0}$  as a left  $KH$ -module via the representation mentioned in (6.1), we claim that  $T$  is balanced. In fact, for every  $k \in \mathcal{S}$ ,  $h \in H$ , and  $\xi$  in  $V_{x_0}$ , one has

$$T(\delta_k \delta_h, \xi) = T(\delta_{kh}, \xi) = U_{kh}(\xi) \stackrel{(5.10)}{=} U_k(U_h(\xi)) = T(\delta_k, U_h(\xi)) = T(\delta_k, \delta_h \cdot \xi).$$

Therefore there exists a unique linear map  $\tau : M \otimes V_{x_0} \rightarrow W_{x_0}$ , such that  $\tau(\delta_k \otimes \xi) = U_k(\xi)$ . We shall next prove that  $\tau$  is an isomorphism by exhibiting an inverse for it.

With this goal in mind, let  $R$  be a system of representatives of left classes for  $\mathcal{S}$  modulo  $H$ . Thus, if  $x$  is in the orbit of  $x_0$ , there exists a unique  $r$  in  $R$  such that  $\theta_r(x_0) = x$ , so that  $U_{r^{-1}}$  maps  $V_x$  onto  $V_{x_0}$ , by (5.10). We therefore let

$$\sigma_x : V_x \rightarrow M \otimes V_{x_0}$$

be given by  $\sigma_x(\xi) = \delta_r \otimes U_{r^{-1}}(\xi)$ , for every  $\xi$  in  $V_x$ . Putting all of the  $\sigma_x$  together, let

$$\sigma : W_{x_0} = \bigoplus_{x \in \text{Orb}(x_0)} V_x \longrightarrow M \otimes V_{x_0}$$

be the only linear map coinciding with  $\sigma_x$  on  $V_x$ , for every  $x$  in  $\text{Orb}(x_0)$ .

We claim that  $\sigma$  is the inverse of  $\tau$ . To see this, let  $k$  be any element of  $\mathcal{S}$ , and let  $\xi$  be picked at random in  $V_{x_0}$ . Writing  $k = rh$ , with  $r \in R$ , and  $h \in H$ , set  $x = \theta_k(x_0) = \theta_r(x_0)$ , so  $U_k(\xi) \in V_x$ . We then have

$$\sigma(\tau(\delta_k \otimes \xi)) = \sigma(U_k(\xi)) = \delta_r \otimes U_{r^{-1}}(U_k(\xi)) = \delta_r \otimes U_h \xi = \delta_{rh} \otimes \xi = \delta_k \otimes \xi.$$

This proves that  $\sigma\tau$  is the identity on  $M \otimes V_{x_0}$ . On the other hand, given any  $x$  in  $\text{Orb}(x_0)$ , and any  $\xi \in V_x$ , write  $x = \theta_r(x_0)$ , with  $r \in R$ , and notice that

$$\tau(\sigma(\xi)) = \tau(\delta_r \otimes U_{r^{-1}}(\xi)) = U_r(U_{r^{-1}}(\xi)) = \xi,$$

so we see that  $\tau\sigma$  is the identity on  $W_{x_0}$ .

Therefore  $\tau$  is an isomorphism between the  $K$ -vector spaces  $M \otimes V_{x_0}$  and  $W_{x_0}$ . We will next prove that  $\tau$  is covariant for the respective actions of  $\mathcal{L}_c(X) \rtimes G$ , which amount to saying that it is linear as a map between left  $(\mathcal{L}_c(X) \rtimes G)$ -modules. For this, given  $g \in G$ , and  $f \in \mathcal{L}_c(X_g)$ , we must prove that

$$\tau((f\Delta_g)\delta_k \otimes \xi) = \rho(f\Delta_g)(\tau(\delta_k \otimes \xi)), \quad \forall k \in \mathcal{S}, \quad \forall \xi \in V_{x_0}. \quad (6.2.1)$$

Given  $k$  and  $\xi$  as indicated above, the left-hand-side equals

$$\tau((f\Delta_g)\delta_k \otimes \xi) = [gk \in \mathcal{S}] f(\theta_{gk}(x_0)) \tau(\delta_{gk} \otimes \xi) = [gk \in \mathcal{S}] f(\theta_{gk}(x_0)) U_{gk}(\xi),$$

while the right-hand-side becomes

$$\rho(f\Delta_g)(\tau(\delta_k \otimes \xi)) = \Pi(f)U_g U_k(\xi) = \cdots \quad (6.2.2)$$

Observe that  $U_k(\xi)$  is in  $V_{\theta_k(x_0)}$ , and recall from (5.10) that  $U_g$  vanishes on  $V_{\theta_k(x_0)}$ , unless  $\theta_k(x_0) \in X_{g^{-1}}$ , in which case  $U_g U_k$  coincides with  $U_{gk}$  on  $V_{x_0}$ . So

$$U_g U_k(\xi) = [\theta_k(x_0) \in X_{g^{-1}}] U_{gk}(\xi).$$

Also notice that

$$\begin{aligned} \theta_k(x_0) \in X_{g^{-1}} &\iff \theta_k(x_0) \in X_{g^{-1}} \cap X_k \iff x_0 \in \theta_{k^{-1}}(X_{g^{-1}} \cap X_k) = X_{k^{-1}g^{-1}} \cap X_{k^{-1}} \iff \\ &\iff x_0 \in X_{k^{-1}g^{-1}} \iff gk \in \mathcal{S}, \end{aligned}$$

where we are taking into account that  $x_0 \in X_{k^{-1}}$  by default. It follows that the expression in (6.2.2) equals

$$\cdots = [gk \in \mathcal{S}] \Pi(f) U_{gk}(\xi) = [gk \in \mathcal{S}] f(\theta_{gk}(x_0)) U_{gk}(\xi),$$

because, in the nonzero case, one has that  $U_{gk}(\xi)$  lies in  $V_{\theta_{gk}(x_0)}$ , and  $\Pi(f)$  acts there by scalar multiplication by  $f(\theta_{gk}(x_0))$ , according to (5.5). This proves (6.2.1), so  $\tau$  is indeed covariant.  $\square$

Summarizing much that we have done so far, the following is the main result of this work:

**6.3. Theorem.** *Let  $\theta = (\{\theta_g\}_{g \in G}, \{X_g\}_{g \in G})$  be a partial action of a discrete group  $G$  on a Hausdorff, locally compact, totally disconnected topological space  $X$ , such that  $X_g$  is clopen for every  $g$  in  $G$ . Then, every ideal  $J \trianglelefteq \mathcal{L}_c(X) \rtimes G$  is the intersection of ideals induced from isotropy groups.*

*Proof.* Let  $R \subseteq X$  be a system of representatives for the orbit relation on  $X$ . Using (5.23) we may write  $J$  as the intersection of the null spaces of the  $\rho_x$ , for  $x$  in  $R$ , while (6.2) tells us that  $\rho_x$  is equivalent to the representation induced from a representation of the isotropy group at  $x$ . The null space of  $\rho_x$  is therefore induced from an ideal in the group algebra of said isotropy group by (3.11), whence the result.  $\square$

Should one want to explicitly write a given ideal  $J \trianglelefteq \mathcal{L}_c(X) \rtimes G$  as the intersection of induced ideals, the next result should come in handy:

**6.4. Proposition.** *Under the assumptions of (6.3), choose a system  $R$  of representatives for the orbit relation on  $X$ . For each  $x$  in  $R$ , let  $H_x$  be the isotropy group at  $x$ , and let*

$$F_x : \mathcal{L}_c(X) \rtimes G \rightarrow KH_x$$

*be as in (3.20). Then, given any ideal  $J \trianglelefteq \mathcal{L}_c(X) \rtimes G$ , one has that  $I'_x := F_x(J)$  is an admissible ideal of  $KH_x$ , and*

$$J = \bigcap_{x \in R} \text{Ind}(I'_x).$$

*Proof.* That each  $I'_x$  is an admissible ideal follows at once from (4.12).

For each  $x$  in  $R$ , let  $I_x$  be the null space of the representation  $\rho_x$  referred to in the proof of (6.3), so that

$$J = \bigcap_{x \in R} \text{Ind}(I_x).$$

Observe that for each  $x \in R$ , one has

$$I'_x = F_x(J) = F_x\left(\bigcap_{y \in R} \text{Ind}(I_y)\right) \subseteq F_x(\text{Ind}(I_x)) \stackrel{(4.5)}{\subseteq} I_x.$$

Consequently  $\text{Ind}(I'_x) \subseteq \text{Ind}(I_x)$ , whence

$$\bigcap_{x \in R} \text{Ind}(I'_x) \subseteq \bigcap_{x \in R} \text{Ind}(I_x) = J.$$

On the other hand, one has by (4.11) that  $\text{Ind}(I'_x)$  is the largest among the ideals of  $\mathcal{L}_c(X) \rtimes G$  mapping into  $I'_x$  under  $F_x$ . Since  $F_x(J) = I'_x$ , by definition, we have that  $J$  is among such ideals, so  $J \subseteq \text{Ind}(I'_x)$ , and then

$$J \subseteq \bigcap_{x \in R} \text{Ind}(I'_x),$$

concluding the proof.  $\square$

## 7. Primitive, prime and meet-irreducible ideals.

Recall that an ideal  $J$  in an algebra  $A$  is said to be *primitive* if it coincides with the annihilator of some irreducible module. It is called *prime* if, whenever  $K$  and  $L$  are ideals in  $A$ , then

$$KL \subseteq J \Rightarrow (K \subseteq J) \vee (L \subseteq J).$$

Finally,  $J$  is said to be *meet-irreducible* if, for any ideals  $K$  and  $L$  in  $A$ , one has

$$K \cap L \subseteq J \Rightarrow (K \subseteq J) \vee (L \subseteq J).$$

It is well known that every primitive ideal is prime, and since the inclusion “ $KL \subseteq K \cap L$ ” holds for any ideals  $K$  and  $L$ , it is clear that every prime ideal is meet-irreducible.

The main goal of this section is to show that the induction process preserves all of the properties mentioned above.

As usual we continue working under (2.1).

**7.1. Proposition.** *If  $I$  is a primitive ideal of  $KH$ , then  $\text{Ind}(I)$  is primitive.*

*Proof.* By hypothesis  $I$  is the annihilator of some irreducible  $KH$ -module  $V$ . Employing (3.11) we then have that  $\text{Ind}(I)$  is the annihilator of the induced module  $M \otimes V$ , which is irreducible by (3.7). Thus  $\text{Ind}(I)$  is primitive.  $\square$

In order to deal with primeness and meet-irreducibility, we first need to prove a technical result:

**7.2. Lemma.** *Let  $J$  and  $K$  be ideals in  $\mathcal{L}_c(X) \rtimes G$ . Then*

- (i)  $F(J) \cap F(K) \subseteq F(J \cap K)$ .
- (ii)  $F(J)F(K) \subseteq F(JK)$ .

*Proof.* We begin by proving (i). For this, let

$$c = \sum_{h \in \Gamma} c_h \delta_h \in F(J) \cap F(K),$$

where  $\Gamma$  is a finite subset of  $H$ . Applying (4.9) twice, we obtain compact-open neighborhoods  $V$  and  $W$  of  $x_0$ , such that  $V, W \subseteq X_h$ , for every  $h \in \Gamma$ , satisfying

$$c_V := \sum_{\Gamma} c_h 1_V \Delta_h \in J \quad \text{and} \quad c_W := \sum_{\Gamma} c_h 1_W \Delta_h \in K.$$

Setting  $Z = V \cap W$ , we have that  $Z$  is another compact-open neighborhood of  $x_0$ , and

$$J \ni 1_Z c_V = \sum_{\Gamma} c_h 1_Z 1_V \Delta_h = \sum_{\Gamma} c_h 1_Z \Delta_h =: c_Z.$$

A similar reasoning shows that  $c_Z$  also lies in  $K$ , so  $c_Z \in J \cap K$ . Therefore

$$c = F(c_Z) \in F(J \cap K).$$

In order to prove (ii), let  $b \in F(J)$  and  $c \in F(K)$ , and write

$$b = \sum_{h \in \Gamma} b_h \delta_h, \quad \text{and} \quad c = \sum_{h \in \Gamma} c_h \delta_h,$$

where  $\Gamma$  is a finite subset of  $H$ . By (4.9), there are compact-open sets  $V$  and  $W$ , such that  $x_0 \in V, W \subseteq X_h$ , for every  $h \in \Gamma$ , satisfying

$$b_V := \sum_{\Gamma} b_h 1_V \Delta_h \in J \quad \text{and} \quad c_W := \sum_{\Gamma} c_h 1_W \Delta_h \in K.$$

Observing that  $b_V$  and  $c_W$  lie in  $\mathcal{L}_c(X) \rtimes H$ , we then have that

$$\begin{aligned} bc &= F(b_V)F(c_W) = \nu(E(b_V))\nu(E(c_W)) = \nu(b_V)\nu(c_W) = \nu(b_V c_W) = \\ &= \nu(E(b_V c_W)) = F(b_V c_W) \in F(JK). \end{aligned}$$

$\square$

We may now prove the result announced earlier:

**7.3. Theorem.** *Let  $I$  be an ideal in  $KH$ . If  $I$  is prime or meet-irreducible, then so is  $\text{Ind}(I)$ .*

*Proof.* Let us first address meet-irreducibility, so suppose that  $K$  and  $L$  are ideals in  $\mathcal{L}_c(X) \rtimes G$ , such that  $K \cap L \subseteq \text{Ind}(I)$ . Then

$$F(K) \cap F(L) \stackrel{(7.2)}{\subseteq} F(K \cap L) \subseteq F(\text{Ind}(I)) \stackrel{(4.5.ii)}{\subseteq} I.$$

Assuming that  $I$  is meet-irreducible, we have that either  $F(K)$  or  $F(L)$  is contained in  $I$ . Supposing without loss of generality that the first alternative is true, that is,  $F(K) \subseteq I$ , we then have

$$K \stackrel{(4.11.i)}{\subseteq} \text{Ind}(F(K)) \subseteq \text{Ind}(I),$$

so  $\text{Ind}(I)$  is meet-irreducible. The proof of the result for prime ideals is obtained by going through the present proof, replacing all intersections with products.  $\square$

## 8. Topologically free points.

As we already hinted upon, topologically free minimal actions prevent the appearance of nontrivial induced ideals. In this section we wish to further explore this aspect. We keep enforcing (2.1).

### 8.1. Definition.

- (i) We say that  $\theta$  is a *topologically free* partial action if, for every  $g$  in  $G \setminus \{1\}$ , the fixed point set

$$F_g := \{x \in X_{g^{-1}} : \theta_g(x) = x\}$$

has empty interior.

- (ii) We shall say that a point  $x_0$  in  $X$  is *topologically free* if, for every  $g$  in  $G \setminus \{1\}$ , and every open set  $V$ , with  $x_0 \in V \subseteq X_{g^{-1}}$ , there exists some  $y \in V \cap \text{Orb}(x_0)$ , such that  $\theta_g(y) \neq y$ .

If  $x_0$  is not fixed by  $\theta_g$ , then the point  $y$  referred to in (8.1.ii) may clearly be taken to be  $x_0$  itself, so the condition is automatically satisfied for such a  $g$ . In other words, this condition is only relevant for  $g$  in the isotropy group of  $x_0$ .

Another way to describe the notion of topologically free point is to say that there is no subset of  $\text{Orb}(x_0)$  containing  $x_0$ , open in the relative topology, and consisting of fixed points for a nontrivial group element  $g$ .

Given the relative notion of the concept of “interior”, one may find a topologically free partial action admitting a invariant subspace  $Y \subseteq X$ , such that the restriction of  $\theta$  to  $Y$  is no longer topologically free. However it is clear that the notion of topologically free *point* is not affected by restricting the action to an invariant subset, as long as the point under consideration lies in such a subset.

Still another equivalent description of topologically free points is given by the following:

**8.2. Proposition.** *Given  $x_0$  in  $X$ , the following are equivalent:*

- (i)  $x_0$  is topologically free,  
(ii) the restriction of  $\theta$  to  $\overline{\text{Orb}(x_0)}$  (the closure of the orbit of  $x_0$ ) is a topologically free partial action.

*Proof.* (i)  $\Rightarrow$  (ii). Since we are only concerned with  $\overline{\text{Orb}(x_0)}$ , rather than the whole of  $X$ , we may replace the latter by the former, and hence assume that the orbit of  $x_0$  is dense in  $X$ . As already observed, this restriction does not affect condition (i).

Assume by contradiction that  $g$  is a nontrivial group element whose fixed point set  $F_g$  has a nontrivial interior, so there exists a nonempty open set  $V \subseteq F_g$ . Since the orbit of  $x_0$  is assumed to be dense, there is some  $k$  in  $\mathcal{S}$  such that  $\theta_k(x_0) \in V$ . It is then easy to prove that  $\theta_{k^{-1}gk}$  is the identity on the open set

$$U := \theta_{k^{-1}}(X_k \cap V),$$

which contains  $x_0$ . In particular  $\theta_{k^{-1}gk}$  is the identity on  $U \cap \text{Orb}(x_0)$ , hence contradicting (i).

(ii)  $\Rightarrow$  (i). Again by contradiction, assume that  $1 \neq g \in G$ , and that  $V$  is an open set with  $x_0 \in V \subseteq X_{g^{-1}}$ , and  $V \cap \text{Orb}(x_0) \subseteq F_g$ . We then claim that

$$V \cap \overline{\text{Orb}}(x_0) \subseteq F_g,$$

as well. To see this, let  $y \in V \cap \overline{\text{Orb}}(x_0)$ . Then  $y$  is the limit of a net  $\{u_i\}_i \subseteq \text{Orb}(x_0)$ , and since  $y \in V$ , we have that  $u_i \in V$  for all sufficiently large  $i$ . For such  $i$ 's we have  $u_i \in V \cap \text{Orb}(x_0) \subseteq F_g$ , so

$$\theta_g(y) = \lim_i \theta_g(u_i) = \lim_i u_i = y,$$

proving that  $y \in F_g$ . The claim is therefore proven, contradicting (ii).  $\square$

As a consequence we see that two points in  $X$  having the same orbit closure are either both topologically free or both fail to satisfy this property.

Topologically free points actually enjoy a slightly stronger property as described next:

**8.3. Proposition.** *Let  $x_0$  be a topologically free point, let  $\Gamma$  be a finite subset of  $G \setminus \{1\}$ , and let  $V$  be an open set with*

$$x_0 \in V \subseteq \bigcap_{g \in \Gamma} X_{g^{-1}}.$$

*Then there exists some  $y \in V \cap \text{Orb}(x_0)$ , such that  $\theta_g(y) \neq y$ , for all  $g$  in  $\Gamma$ .*

*Proof.* By restricting  $\theta$  to the closure of the orbit of  $x_0$  we may assume that  $\text{Orb}(x_0)$  is dense in  $X$ .

For each  $g$  in  $\Gamma$ , let

$$\Phi_g = F_g \cap V = \{x \in V : \theta_g(x) = x\}.$$

Then clearly  $\Phi_g$  is a closed subset (relative to  $V$ ) and by (8.2) we have that  $\Phi_g$  has no interior (relative to  $X$ , and hence also relative to  $V$ ). Consequently  $\bigcup_{g \in \Gamma} \Phi_g$  is a closed set with empty interior<sup>9</sup>, whence

$$V \setminus \bigcup_{g \in \Gamma} \Phi_g$$

is a nonempty open set (relative to  $V$  and hence also relative to  $X$ ). Since we are assuming that the orbit of  $x_0$  is dense, we conclude that there is some  $y$  in said orbit which also lies in the above open set. This concludes the proof.  $\square$

The conflict between topological freeness and induced ideals is clearly expressed by the following:

**8.4. Proposition.** *When  $x_0$  is topologically free, the only admissible ideals of  $KH$  are the trivial ones, namely  $\{0\}$  and  $KH$ , itself. Consequently the only induced ideals arising from ideals in  $KH$  are the trivial ones described in (3.15).*

*Proof.* Let  $I \trianglelefteq KH$  be a nonzero admissible ideal. We first claim that there exists some  $c = \sum_{h \in H} c_h \delta_h \in I$ , with  $c_1 \neq 0$ . To see this let  $d = \sum_{h \in H} d_h \delta_h$  be any nonzero element of  $I$ . Choose  $h_0$  in  $H$  such that  $d_{h_0} \neq 0$ , and let

$$c = d\delta_{h_0^{-1}} = \sum_{h \in H} d_h \delta_{hh_0^{-1}},$$

so that  $c$  is also in  $I$ , and  $c_1 = d_{h_0} \neq 0$ , proving the claim.

Working with  $c$ , choose  $V$  as in (4.10). Letting

$$\Gamma = \{h \in H : c_h \neq 0\},$$

we may clearly assume that  $V \subseteq \bigcap_{h \in \Gamma} X_{h^{-1}}$ . Employing (8.3), let  $y$  be an element of the orbit of  $x_0$ , belonging to  $V$ , and not fixed by any  $h \in \Gamma \setminus \{1\}$ .

---

<sup>9</sup> A finite union of closed sets with empty interior always has empty interior.

Writing  $y = \theta_k(x_0)$ , we claim that  $\Gamma \cap kHk^{-1} = \{1\}$ . To see this it is enough to observe that  $\theta_k(x_0)$  is fixed by the elements of  $kHk^{-1}$ , while the only element of  $\Gamma$  having this property is the unit.

By (4.10) we then conclude that

$$I \ni \delta_{k^{-1}} \left( \sum_{h \in \Gamma \cap kHk^{-1}} c_h \delta_h \right) \delta_k = \delta_{k^{-1}} (c_1 \delta_1) \delta_k = c_1 \delta_1.$$

This implies that  $I$  contains a nonzero multiple of the unit  $\delta_1$ , an invertible element, whence  $I = KH$ , concluding the proof.

Regarding the last sentence in the statement, if  $I$  is any ideal in  $KH$ , then by (4.7) there exists an admissible ideal  $I'$  such that  $\text{Ind}(I) = \text{Ind}(I')$ . By the first part of the proof we have that  $I'$  is either  $\{0\}$  or  $KH$ , as desired.  $\square$

## 9. Regular Points.

In this section, still under (2.1), we will study points possessing a property which may be considered as being in the other end of the spectrum, relative to topological freeness.

**9.1. Definition.** A point  $x_0$  in  $X$  is said to be *strongly regular* (resp. *regular*) if, for every  $h$  in the isotropy group of  $x_0$ , there exists an open set  $V$  with  $x_0 \in V \subseteq X_{h^{-1}}$ , and such that  $\theta_h$  is the identity on  $V$  (resp. on  $V \cap \text{Orb}(x_0)$ ).

Like the notion of topological free point, the notion of regular point given above is phrased in such a way as to depend only on the action of  $G$  on the orbit of the point under consideration (seen under the relative topology). This has the advantage of being mostly an attribute of the point, rather than of the action. However, the same cannot be said of the notion of strongly regular point. In any case, it is easy to see that every strongly regular point is also regular.

The following result is the partial actions version of (and it follows from) [2: Lemma 3.3.a].

**9.2. Proposition.** *If  $G$  is countable, then the set of strongly regular points is dense.*

*Proof.* Observe that a point  $x_0$  fails to be strongly regular precisely when it lies in the fixed point set  $F_g$  for some  $g$  in  $G$ , but it does not belong to the interior of  $F_g$ . This is obviously to say that  $x_0 \in \partial F_g$ , meaning the boundary of  $F_g$ . So the set of points which are not strongly regular is precisely the set

$$\mathcal{S} := \bigcup_{g \in G} \partial F_g.$$

On the other hand, since  $F_g$  is closed, its boundary is a closed set with empty interior. Therefore, should  $G$  be countable, we have that  $\mathcal{S}$  is of first category in Baire's sense, hence its complement, namely the set of strongly regular points, is dense.  $\square$

For regular points, a much simpler characterization of admissibility may be given, if compared to (4.10). This will be done based on a simpler decoding of the information that " $c_V \in \text{Ind}(I)$ " in the first paragraph of the proof of (4.10). In order to highlight this simplification, which will be used elsewhere later, we will isolate the technicalities involved in the next two auxiliary results.

**9.3. Lemma.** *Let  $\Gamma$  be any subset of  $G$ , and let  $k \in \mathcal{S}$  be such that  $\theta_k(x_0)$  is fixed by  $\theta_g$ , for all  $g$  in  $\Gamma$ . Then*

- (i)  $\Gamma \subseteq kHk^{-1}$ ,
- (ii) *for every  $l$  in  $G$  such that  $\Gamma \cap kHl^{-1}$  is nonempty, one has that  $l \in \mathcal{S}$ , that  $\theta_l(x_0) = \theta_k(x_0)$ , and moreover  $\Gamma \subseteq kHl^{-1}$ .*

*Proof.* (i) Given  $g \in \Gamma$ , we have that  $\theta_g(\theta_k(x_0)) = \theta_k(x_0)$ , so  $\theta_{k^{-1}}(\theta_g(\theta_k(x_0))) = x_0$ , whence  $k^{-1}gk$  is in  $H$ , and consequently  $g \in kHk^{-1}$ . This proves (i).

(ii) Pick  $g_1$  in  $\Gamma \cap kHl^{-1}$ , so that  $h_1 := k^{-1}g_1l \in H$ . Therefore

$$\theta_k(x_0) = \theta_{g_1^{-1}}(\theta_k(x_0)) = \theta_{g_1^{-1}}(\theta_k(\theta_{h_1}(x_0))) = \theta_{g_1^{-1}kh_1}(x_0) = \theta_l(x_0).$$

This proves that  $l$  is in  $\mathcal{S}$ , and that  $\theta_k(x_0) = \theta_l(x_0)$ . Next, picking any  $g \in \Gamma$ , notice that

$$x_0 = \theta_{k^{-1}}(\theta_k(x_0)) = \theta_{k^{-1}}(\theta_g(\theta_k(x_0))) = \theta_{k^{-1}}(\theta_g(\theta_l(x_0))) = \theta_{k^{-1}gl}(x_0).$$

Thus  $k^{-1}gl \in H$ , and so  $g \in kHl^{-1}$ , proving (ii). □

**9.4. Lemma.** *Let  $I$  be an ideal in  $KH$ , and let  $c = \sum_{g \in \Gamma} c_g \delta_g$ , be an arbitrary element of  $KG$ , where  $\Gamma$  is a finite subset of  $G$ . Suppose that  $V$  is a compact-open set such that  $V \subseteq X_g$ , and  $\theta_{g^{-1}}$  coincides with the identity on  $V \cap \text{Orb}(x_0)$ , for all  $g$  in  $\Gamma$ . Then*

$$c_V := \sum_{g \in \Gamma} c_g 1_V \Delta_g$$

*lies in  $\text{Ind}(I)$  if and only if, for every  $k$  in  $\mathcal{S}$ , such that  $\theta_k(x_0) \in V$ , one has that*

$$\delta_{k^{-1}} c \delta_k \in I.$$

*Proof.* We begin with the “only if” part, so we assume that  $c_V \in \text{Ind}(I)$ . Given  $k$  in  $\mathcal{S}$ , with  $\theta_k(x_0) \in V$ , we have

$$I \ni \langle \delta_k, c_V \delta_k \rangle \stackrel{(3.13)}{=} \sum_{g \in \Gamma \cap kHk^{-1}} c_g 1_V(\theta_k(x_0)) \delta_{k^{-1}gk} = \delta_{k^{-1}} \left( \sum_{g \in \Gamma \cap kHk^{-1}} c_g \delta_g \right) \delta_k.$$

Observing that  $\theta_k(x_0)$  lies in  $V \cap \text{Orb}(x_0)$ , we have by hypotheses that  $\theta_k(x_0)$  is fixed by  $\theta_{g^{-1}}$ , and hence also by  $\theta_g$ , for every  $g$  in  $\Gamma$ . We then conclude from (9.3.i) that  $\Gamma \subseteq kHk^{-1}$ , so the computation above gives  $\delta_{k^{-1}} c \delta_k \in I$ , as desired.

In order to prove the “if” part, let us show that  $c_V$  lies in  $\text{Ind}(I)$  by employing the criteria given in (3.14). For this we must prove that, for every  $k$  and  $l$  in  $\mathcal{S}$ , one has that

$$\sum_{g \in \Gamma \cap kHl^{-1}} c_g 1_V(\theta_k(x_0)) \delta_{k^{-1}gl} \in I. \quad (9.4.1)$$

There are two situations in which the above vanishes, in which case there is nothing to do, namely when  $\Gamma \cap kHl^{-1}$  is the empty set, or when  $\theta_k(x_0) \notin V$ . Ignoring these, let us assume that the opposite is true, namely that  $\Gamma \cap kHl^{-1}$  is nonempty and that  $\theta_k(x_0)$  lies in  $V$ , which in turn implies that  $\theta_k(x_0)$  is fixed by  $\Gamma$ . Therefore (9.3.ii) gives  $\theta_k(x_0) = \theta_l(x_0)$ , and  $\Gamma \subseteq kHl^{-1}$ . We then see that the term appearing in (9.4.1) is given by

$$\sum_{g \in \Gamma} c_g \delta_{k^{-1}gl} = \delta_{k^{-1}} \left( \sum_{g \in \Gamma} c_g \delta_g \right) \delta_l = \delta_{k^{-1}} c \delta_l = \delta_{k^{-1}} c \delta_k \delta_{k^{-1}l}.$$

To see that this lies in  $I$ , notice that  $k^{-1}l \in H$ , because  $\theta_k(x_0) = \theta_l(x_0)$ , and moreover that  $\delta_{k^{-1}} c \delta_k$  is in  $I$  by hypothesis. So (9.4.1) follows from the fact that  $I$  is an ideal in  $KH$ . We then conclude that  $c_V \in \text{Ind}(I)$ , thanks to (3.14). □

The promised simplified characterization of admissibility is given next:

**9.5. Proposition.** *Suppose that  $x_0$  is regular. Then an ideal  $I \trianglelefteq KH$  is admissible if and only if, for every  $c$  in  $I$ , there exists a neighborhood  $V$  of  $x_0$ , such that*

$$\delta_{k^{-1}} c \delta_k \in I,$$

*for all  $k$  in  $\mathcal{S}$ , such that  $\theta_k(x_0) \in V$ .*



*Proof.* We begin exactly as in the proof of (4.10): supposing that  $I$  is admissible, pick  $c = \sum_{h \in \Gamma} c_h \delta_h$  in  $I$ , where  $\Gamma$  is a finite subset of  $H$ . By hypothesis  $c$  is in  $F(\text{Ind}(I))$ , so (4.9) provides a compact-open neighborhood  $V$  of  $x_0$ , such that

$$c_V := \sum_{h \in \Gamma} c_h 1_V \Delta_h \in \text{Ind}(I).$$

Given that  $x_0$  is regular, and upon shrinking  $V$ , if necessary, we may assume that  $\theta_{h^{-1}}$  is the identity on  $V \cap \text{Orb}(x_0)$ , for every  $h$  in  $\Gamma$ . The conclusion then follows from (9.4).

Conversely, assuming that  $I$  satisfies the condition in the statement, let us prove that  $I$  is admissible, namely that  $F(\text{Ind}(I)) \supseteq I$ . So, pick any  $c = \sum_{h \in \Gamma} c_h \delta_h$  in  $I$ , where  $\Gamma$  is a finite subset of  $H$ . Using the hypothesis, we then choose  $V$  as in the statement, which we may clearly suppose to be compact-open. Again because  $x_0$  is regular, we may assume that  $\theta_{h^{-1}}$  is defined and coincides with the identity on  $V \cap \text{Orb}(x_0)$ , for every  $h$  in  $\Gamma$ .

By hypothesis, and by (9.4), it follows that  $c_V \in \text{Ind}(I)$ , whence

$$c = F(c_V) \in F(\text{Ind}(I)),$$

as desired. □

An important, albeit trivial conclusion to be drawn from the above result is:

**9.6. Corollary.** *If  $G$  is commutative and  $x_0$  is a regular point of  $X$ , then every ideal of  $KH$  is admissible.*

## 10. Normal ideals.

It is interesting to notice that, while the admissibility condition given in (9.5) is a combination of dynamical features (viz. “ $\theta_k(x_0) \in V$ ”) and algebraic properties (viz. “ $\delta_{k^{-1}} c \delta_k \in I$ ”), the algebraic properties alone ensure admissibility in (9.6).

In this section we shall discuss other purely algebraic conditions on an ideal of  $KH$  which are enough to guarantee admissibility, regardless of any other dynamical restrictions.

Given a group  $G$  and a field  $K$ , recall that the well known *adjoint* action of  $G$  on  $KG$  is the map

$$\text{Ad} : G \rightarrow \text{Aut}(KG)$$

given by

$$\text{Ad}_g(a) = \delta_g a \delta_{g^{-1}}, \quad \forall g \in G, \quad \forall a \in KG.$$

Given any subgroup  $H$  of  $G$ , observe that  $KH$  is invariant under  $\text{Ad}$  if and only if  $H$  is a normal subgroup. Regardless of normality, we may always restrict  $\text{Ad}$  to a *partial action* of  $G$  on  $KH$ , as in [4: 3.2]. The main ingredients of this construction are as follows: for each  $g$  in  $G$ , we let

$$D_g = KH \cap \text{Ad}_g(KH),$$

and we let

$$p\text{Ad}_g : D_{g^{-1}} \rightarrow D_g$$

be the restriction of  $\text{Ad}_g$  to  $D_{g^{-1}}$ . It is well known that  $p\text{Ad}$  is then a partial action (in the category of sets).

**10.1. Definition.** The above partial action will be called the *adjoint partial action* of  $G$  on  $KH$ .

It is easy to see that each  $D_g$  is a subalgebra of  $KH$ , while the  $p\text{Ad}_g$  are algebra isomorphisms. However  $p\text{Ad}$  cannot be viewed as an algebraic partial action, as defined in [4: 6.4], because the  $D_g$  are not ideals in  $KH$ , but alas,  $p\text{Ad}$  is a legitimate set theoretical partial action cf. [4: 2.1].

**10.2. Definition.** Let  $H$  be a subgroup of a group  $G$ , and let  $I$  be an ideal in  $KH$ . We shall say that  $I$  is *normal relative to  $G$* , if  $I$  is invariant [4: 2.9] under the adjoint partial action of  $G$  on  $KH$ .

Thus, to say that  $I$  is normal is to say that for every  $c \in I$ , and every  $g$  in  $G$  such that  $\delta_g c \delta_{g^{-1}} \in KH$ , one has that  $\delta_g c \delta_{g^{-1}} \in I$ .

One should view this as the best possible effort made by the ideal  $I$  in trying to embrace all element of the above form  $\delta_g c \delta_{g^{-1}}$ , except of course that this is impossible in the hopeless cases when such elements are not even in  $KH$ !

**10.3. Proposition.** *Under (2.1), let  $x_0$  be a regular point of  $X$ , and let  $H$  be its isotropy group. Then every ideal  $I \trianglelefteq KH$  which is normal relative to  $G$ , is also admissible.*

*Proof.* We will verify the conditions of (9.5). Thus, given  $c$  in  $I$ , write  $c = \sum_{h \in \Gamma} c_h \delta_h$ , where  $\Gamma \subseteq H$  is a finite set, and choose a neighborhood  $V$  of  $x_0$ , such that  $\theta_h$  is the identity map on  $V \cap \text{Orb}(x_0)$ , for every  $h$  in  $\Gamma$ . Still focusing on (9.5), pick any  $k$  in  $\mathcal{S}$  such that  $\theta_k(x_0) \in V$ .

We then claim that  $\delta_{k^{-1}} c \delta_k \in KH$ . To see this, notice that, for every  $h$  in  $\Gamma$ , one has that  $\theta_h$  fixes  $\theta_k(x_0)$ , meaning that  $\theta_h(\theta_k(x_0)) = \theta_k(x_0)$ , from where we deduce that

$$\theta_{k^{-1}}(\theta_h(\theta_k(x_0))) = x_0.$$

So  $k^{-1} h k \in H$ , whence

$$\delta_{k^{-1}} c \delta_k = \sum_{h \in \Gamma} c_h \delta_{k^{-1} h k} \in KH.$$

The invariance of  $I$  under the adjoint partial action then implies that  $\delta_{k^{-1}} c \delta_k \in I$ , concluding the verification of the conditions of (9.5), and hence proving that  $I$  is admissible.  $\square$

A source of examples of normal ideals is as follows:

**10.4. Proposition.** *Let  $H$  be a subgroup of a group  $G$ , and let  $J$  be any ideal in  $KG$ . Then the ideal  $I$  of  $KH$  given by*

$$I = J \cap KH$$

*is normal relative to  $G$ .*

*Proof.* If  $c$  is in  $I$ , then for every  $g$  in  $G$ , one has that  $\delta_g c \delta_{g^{-1}} \in J$ . If the latter happens to also lie in  $KH$ , then it clearly belongs to  $I$ . Therefore  $I$  is normal.  $\square$

A concrete example is the *augmentation ideal*  $I_H$  given by

$$I_H = \text{Ker}(\varepsilon_H,)$$

where  $\varepsilon_H$  is the *augmentation map*, namely the map  $\varepsilon_H : KH \rightarrow K$ , given by

$$\varepsilon_H\left(\sum_{h \in H} c_h \delta_h\right) = \sum_{h \in H} c_h.$$

**10.5. Proposition.** *Let  $H$  be a subgroup of a group  $G$ . Then the augmentation ideal  $I_H$  is normal relative to  $G$ .*

*Proof.* This ideal being the intersection of  $KH$  with the augmentation ideal  $I_G$  of  $G$ , the conclusion follows from (10.4).  $\square$

Incidentally, the ideal referred to in [1] is related to the ideal induced by  $I_H$ . In particular we have:

**10.6. Proposition.** (cf. [1]) *Assuming (2.1) and that  $\mathcal{L}_c(X) \rtimes G$  is simple, one has that  $\theta$  is topologically free.*

*Proof.* Suppose by contradiction that  $\theta$  is not topologically free. Then there exists a nontrivial  $g$  in  $G$  whose fix point set  $F_g$  has nonempty interior. Since the regular points are dense in  $X$ , we may pick a regular point  $x_0$  in  $V$ . In particular  $g$  lies in the isotropy group  $H$  of  $x_0$ , so  $H$  is a nontrivial group, whence

$$\{0\} \subsetneq I_H \subsetneq KH,$$

where  $I_H$  is the augmentation ideal of  $KH$ . Observe that the three ideals above are admissible by (4.8), (10.5) and (10.3), so by the uniqueness part of (4.7), we have

$$\text{Ind}(\{0\}) \subsetneq \text{Ind}(I_H) \subsetneq \text{Ind}(KH).$$

However, since  $\mathcal{L}_c(X) \rtimes G$  is supposed to be a simple algebra, it is impossible to find three distinct ideals as above. This is a contradiction, and hence the statement is proved.  $\square$

In view of (10.3) one could ask whether conditions can be found regarding an ideal  $I \trianglelefteq KH$ , which would ensure  $I$  to be admissible regardless of any dynamical condition, as in (10.3), but also regardless of  $x_0$  being a regular point. Except for the trivial ideals treated in (3.15), this seems to be impossible in view of (8.4), where topological freeness, an eminently dynamical condition, overrides any algebraic condition one could think of.

### 11. Transposition.

So far we have concentrated our study on induced ideals relative to a single point  $x_0$  in  $X$ , but now we would like to conduct a comparative study. So, besides assuming (2.1), and hence having fixed a point  $x_0$ , we will fix another point in  $X$ , denoted  $\hat{x}_0$ , and we will discuss the relationship between ideals induced relative to  $x_0$  and its peer  $\hat{x}_0$ .

Having two points in sight, it is now crucial that we distinguish the sets  $H$  and  $\mathcal{S}$  introduced in (3.1), depending on whether  $x_0$  or  $\hat{x}_0$  is concerned. One alternative would be to employ their official notation with corresponding subscripts, such as “ $H_{x_0}$ ”, “ $\mathcal{S}_{x_0}$ ”, “ $H_{\hat{x}_0}$ ” and “ $\mathcal{S}_{\hat{x}_0}$ ”. However we will really only consider the induction process for the two points  $x_0$  and  $\hat{x}_0$  chosen above, so we will prefer to save on notation by keeping the undecorated notation when  $x_0$  is considered, and writing  $\hat{H}$  and  $\hat{\mathcal{S}}$ , when we are talking about  $\hat{x}_0$ .

The maps  $E$ ,  $\nu$  and  $F$ , respectively introduced in (3.17), (3.19) and (3.20), also need to be distinguished, so we will adopt the above policy of decorating everything regarding  $\hat{x}_0$  with a “hat”.

Finally, the induction process itself needs to be distinguished, so we will write  $\hat{Ind}(\hat{I})$ , if inducing an ideal  $\hat{I} \trianglelefteq K\hat{H}$ , relative to  $\hat{x}_0$ , while retaining our previous notation regarding  $x_0$ .

The crucial way in which the two induction processes are related may be subsumed by a correspondence between ideals in  $KH$  and ideals in  $K\hat{H}$ , defined as follows: given an ideal  $I \trianglelefteq KH$ , we may form the induced ideal  $Ind(I)$ , and then we have by (4.12) that  $\hat{F}(Ind(I))$  is an admissible ideal in  $K\hat{H}$  (relative to  $\hat{x}_0$ , of course).

**11.1. Definition.** Given an ideal  $I \trianglelefteq KH$ , we shall let

$$\hat{\mathcal{T}}(I) = \hat{F}(Ind(I)),$$

so that  $\hat{\mathcal{T}}$  is a map from the set of all ideals in  $KH$  into the set of all admissible ideals in  $K\hat{H}$ . We shall refer to  $\hat{\mathcal{T}}(I)$  as the *transposition of  $I$  from  $KH$  to  $K\hat{H}$* . Likewise, given an ideal  $\hat{I} \trianglelefteq K\hat{H}$ , its transposition from  $K\hat{H}$  to  $KH$  is defined by

$$\mathcal{T}(\hat{I}) = F(\hat{Ind}(\hat{I})).$$

Since we are in the business of studying induced ideals we don't really care so much about non admissible ideals, so we will shortly restrict ourselves to transposing admissible ideals only. Nevertheless one might observe that an ideal  $I \trianglelefteq KH$  is admissible if and only if it coincides with its own transposition from  $KH$  to itself.

Even before we fully understand the transposition map, we may prove a few important facts:

**11.2. Proposition.** *Let  $I \trianglelefteq KH$  and  $\hat{I} \trianglelefteq K\hat{H}$  be admissible ideals, then the following are equivalent*

- (i)  $Ind(I) \subseteq \hat{Ind}(\hat{I})$ ,
- (ii)  $\hat{\mathcal{T}}(I) \subseteq \hat{I}$ .

*In addition, when the above equivalent conditions hold, and both  $I$  and  $\hat{I}$  are proper ideals, then  $\overline{Orb}(x_0) \supseteq \overline{Orb}(\hat{x}_0)$ .*

*Proof.* (i)  $\Rightarrow$  (ii): We have

$$\hat{\mathcal{T}}(I) = \hat{F}(Ind(I)) \subseteq \hat{F}(\hat{Ind}(\hat{I})) = \hat{I},$$

where the last equality is a consequence of the fact that  $\hat{I}$  is admissible.

(ii)  $\Rightarrow$  (i): Observing that our hypothesis reads  $\hat{F}(Ind(I)) \subseteq \hat{I}$ , recall from (4.11) that  $\hat{Ind}(\hat{I})$  is the largest ideal mapping into  $\hat{I}$  under  $\hat{F}$ , whence (i) holds.

Regarding the last sentence in the statement, we have by (3.16) that the intersection  $Ind(I) \cap \mathcal{L}_c(X)$  consists of all  $f$  in  $\mathcal{L}_c(X)$  vanishing on  $\overline{Orb}(x_0)$ . Therefore (i) implies that every such  $f$  necessarily also vanishes on  $\overline{Orb}(\hat{x}_0)$ , from where the conclusion follows.  $\square$

The fact that  $\hat{\mathcal{T}}(I) = \hat{I}$  is not equivalent to  $I = \mathcal{T}(\hat{I})$ , so our result for equality of induced ideals must mention both:

**11.3. Theorem.** *Let  $I \trianglelefteq KH$  and  $\hat{I} \trianglelefteq K\hat{H}$  be admissible ideals, then the following are equivalent:*

- (i)  $\text{Ind}(I) = \hat{\text{Ind}}(\hat{I})$ ,
- (ii)  $\hat{\mathcal{T}}(I) = \hat{I}$ , and  $I = \mathcal{T}(\hat{I})$ ,
- (iii)  $\hat{\mathcal{T}}(I) \subseteq \hat{I}$ , and  $I \supseteq \mathcal{T}(\hat{I})$ .

*Proof.* (i)  $\Rightarrow$  (ii): We have

$$\hat{\mathcal{T}}(I) = \hat{F}(\text{Ind}(I)) = \hat{F}(\hat{\text{Ind}}(\hat{I})) \stackrel{(4.6)}{=} \hat{I},$$

and one similarly proves that  $\mathcal{T}(\hat{I}) = I$ .

(ii)  $\Rightarrow$  (iii): Obvious.

(iii)  $\Rightarrow$  (i): Follows immediately from (11.2). □

In order to give a concrete description of a transposed ideal, we must bring in certain important maps between the various group algebras in sight. Initially, consider the natural projection

$$P : \sum_{g \in G} c_g \delta_g \in KG \mapsto \sum_{h \in H} c_h \delta_h \in KH,$$

and, given  $k$  and  $l$  in  $G$ , define the map

$$\Psi_{k,l} : c \in K\hat{H} \mapsto P(\delta_{k^{-1}} c \delta_l) \in KH.$$

For an explicit expression, let  $c = \sum_{h \in \hat{H}} c_h \delta_h \in K\hat{H}$ , and notice that

$$\Psi_{k,l}(c) = P\left(\sum_{h \in \hat{H}} c_h \delta_{k^{-1}hl}\right) = \sum_{\substack{h \in \hat{H} \\ k^{-1}hl \in H}} c_h \delta_{k^{-1}hl} = \delta_{k^{-1}} \left( \sum_{h \in \hat{H} \cap kHl^{-1}} c_h \delta_h \right) \delta_l.$$

**11.4. Proposition.** *Let  $I$  be an admissible ideal in  $KH$ . Then the transposition of  $I$  to  $K\hat{H}$  is given by*

$$\hat{\mathcal{T}}(I) = \bigcup_{V \ni \hat{x}_0} \bigcap_{\substack{k,l \in \mathcal{S} \\ \theta_k(x_0) \in V}} \Psi_{k,l}^{-1}(I),$$

where by “ $V \ni \hat{x}_0$ ” we mean that  $V$  ranges in the family of all neighborhoods of  $\hat{x}_0$ .

*Proof.* Let  $c = \sum_{h \in \hat{H}} c_h \delta_h \in K\hat{H}$ . Then by (4.9) one has that  $c$  lies in  $\hat{\mathcal{T}}(I) = \hat{F}(\text{Ind}(I))$  if and only if there exists a compact-open set  $V$ , such that

$$\hat{x}_0 \in V \subseteq X_h, \tag{11.4.1}$$

whenever  $c_h \neq 0$ , and

$$c_V := \sum_{h \in \hat{H}} c_h 1_V \Delta_h \in \text{Ind}(I).$$

Using (3.14), the above is equivalent to saying that, for every  $k$  and  $l$  in  $\mathcal{S}$ , one has that

$$\begin{aligned} I &\ni \sum_{h \in \hat{H} \cap kHl^{-1}} c_h 1_V(\theta_k(x_0)) \delta_{k^{-1}hl} = \\ &= 1_V(\theta_k(x_0)) \sum_{h \in \hat{H} \cap kHl^{-1}} c_h \delta_{k^{-1}hl} = \\ &= 1_V(\theta_k(x_0)) \Psi_{k,l}(c). \end{aligned}$$

This condition is clearly meaningless unless  $\theta_k(x_0)$  is in  $V$ , in which case it says that  $c \in \Psi_{k,l}^{-1}(I)$ .

It follows that  $c \in \hat{\mathcal{T}}(I)$  if and only if  $c$  lies in the set whose definition is almost exactly what the statement claims  $\hat{\mathcal{T}}(I)$  to be, the only difference being the family of sets where  $V$  ranges which, in the present case, consists of all compact-open neighborhoods of  $\hat{x}_0$  satisfying (11.4.1). However, if we take into account that  $\hat{x}_0$  admits a fundamental system of compact-open neighborhoods, and that the correspondence

$$V \mapsto \bigcap_{\substack{k,l \in \mathcal{S} \\ \theta_k(x_0) \in V}} \Psi_{k,l}^{-1}(I), \quad (11.4.2)$$

is decreasing, then we see that such a difference is irrelevant.  $\square$

An interesting consequence is that, when  $\hat{x}_0$  is not in the closure of the orbit of  $x_0$ , the transposition of ideals leads to a triviality:

**11.5. Proposition.** *Suppose that  $\hat{x}_0 \notin \overline{\text{Orb}}(x_0)$ . Then, for every ideal  $I \trianglelefteq KH$ , one has that  $\hat{\mathcal{T}}(I) = K\hat{H}$ .*

*Proof.* Let  $V$  be a neighborhood of  $\hat{x}_0$  such that  $V \cap \overline{\text{Orb}}(x_0) = \emptyset$ . Then there is no  $k$  in  $\mathcal{S}$  such that  $\theta_k(x_0) \in V$ , whence (11.4.2) consists of the intersection of the empty family of sets, resulting in the universe where it is considered, namely  $K\hat{H}$ .  $\square$

The transposition towards strongly regular points may be described in a much simpler way:

**11.6. Theorem.** *Assume that  $\hat{x}_0$  is strongly regular, and let  $I$  be an admissible ideal in  $KH$ . Then*

$$\hat{\mathcal{T}}(I) = \bigcup_{V \ni \hat{x}_0} \bigcap_{\substack{k \in \mathcal{S} \\ \theta_k(x_0) \in V}} \delta_k I \delta_{k^{-1}}.$$

*Proof.* Let  $c = \sum_{h \in \Gamma} c_h \delta_h \in K\hat{H}$ , where  $\Gamma$  is a finite subset of  $\hat{H}$ . Then by (4.9) we have that the following two conditions are equivalent:

- (i)  $c \in \hat{\mathcal{T}}(I) = \hat{F}(\text{Ind}(I))$ ,
- (ii) there exists a compact-open set  $V$ , such that

$$\hat{x}_0 \in V \subseteq X_h, \quad \forall h \in \Gamma,$$

and

$$c_V := \sum_{h \in \Gamma} c_h 1_V \Delta_h \in \text{Ind}(I).$$

Let us prove that (ii) is in turn equivalent to:

- (iii) there exists a compact-open set  $V$ , satisfying all of the requirements of (ii), and moreover such that  $\theta_h$  fixes  $V$ , for every  $h$  in  $\Gamma$ .

To see that (ii) implies (iii), use the fact that  $\hat{x}_0$  is strongly regular to produce a compact-open neighborhood  $W$  of  $\hat{x}_0$ , such that  $\theta_h$  fixes  $W$ , for every  $h$  in  $\Gamma$ . One then has that

$$\text{Ind}(I) \ni 1_W c_V = \sum_{h \in \Gamma} c_h 1_W 1_V \Delta_h = \sum_{h \in \Gamma} c_h 1_{W \cap V} \Delta_h = c_{W \cap V},$$

thus proving (iii). That (iii) implies (ii) is evident.

Assuming that  $c \in \hat{\mathcal{T}}(I)$ , and hence that (iii) holds, it follows from (9.4) that, for every  $k$  in  $\mathcal{S}$ , with  $\theta_k(x_0) \in V$ , one has that  $\delta_{k^{-1}} c \delta_k \in I$ . Consequently  $c \in \delta_k I \delta_{k^{-1}}$ , which is to say that

$$c \in \bigcap_{\substack{k \in \mathcal{S} \\ \theta_k(x_0) \in V}} \delta_k I \delta_{k^{-1}},$$

which in turn implies that  $c$  belongs to the set the statement claims  $\hat{\mathcal{T}}(I)$  to be.

Conversely, if  $c$  lies in that set, there exists an open neighborhood  $V$  of  $\hat{x}_0$  such that, whenever  $k \in \mathcal{S}$ , and  $\theta_k(x_0) \in V$ , one has that  $c \in \delta_k I \delta_{k^{-1}}$ . Since  $\hat{x}_0$  is strongly regular, and upon shrinking  $V$  if necessary, we may suppose that  $V$  is compact-open, and that  $\theta_h$  fixes  $V$ , for every  $h$  in  $\Gamma$ . It then follows from (9.4) that  $c_V \in \text{Ind}(I)$ , namely that condition (ii) above holds, so that (i) also holds, so  $c \in \hat{\mathcal{T}}(I)$ . This completes the proof.  $\square$

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